5.2: The Natural Logarithmic Function: Integration

$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Using the derivative of the natural logarithmic function to obtain an antiderivative:

Recall: $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Example 1: Find the derivative of $g(x) = \ln|x|$.

Rewrite $g(x)$ without the absolute value.

$g(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

Note: Domain is $(-\infty, 0) \cup (0, \infty)$

$g'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}$

So $\boxed{g'(x) = \frac{1}{x}}$ (except for $x = 0$. Both g and g' are undefined at 0)

Note that $f(x) = \ln x$ has the same derivative as $g(x) = \ln|x|$.

Therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$. This means that $f(x) = \ln|x|$ is an antiderivative of $F(x) = \frac{1}{x}$.

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

Domain $(0, \infty)$

Domain is $(-\infty, 0) \cup (0, \infty)$

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + c}$$

$n \neq -1$

Recall: The power rule for integrals $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ had a restriction: ~~$n \neq 1$~~. Now we can handle this case.

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$

Example 2: Determine $\int \frac{x^2}{x^3+4}\,dx$.

$u = x^3+4$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$\int \frac{x^2}{x^3+4}\,dx = \int \underbrace{x^2 \underbrace{\left(\frac{1}{x^3+4}\right)}_{\frac{1}{u}} dx}_{\frac{1}{3}du} = \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|u| + c$$

$$= \boxed{\frac{1}{3}\ln|x^3+4| + c}$$

Check: $\frac{d}{dx}\left[\frac{1}{3}\ln(x^3+4)\right]$

$$= \frac{1}{\cancel{3}}\cdot\frac{1}{x^3+4}(\cancel{3}x^2) = \frac{x^2}{x^3+4}$$

Example 3: Determine $\int \frac{7}{2-5x}\,dx$.

Example 4: Determine $\int_2^5 \frac{1}{3x}\,dx$.

$$\int_2^5 \frac{1}{3x}\,dx = \frac{1}{3}\int_2^5 \frac{1}{x}\,dx = \frac{1}{3}\ln|x|\Big|_2^5 = \frac{1}{3}\ln|5| - \frac{1}{3}\ln|2|$$

$$= \frac{1}{3}\ln 5 - \frac{1}{3}\ln 2$$

$$= \frac{1}{3}(\ln 5 - \ln 2)$$

$$= \boxed{\frac{1}{3}\ln\left(\frac{5}{2}\right)}$$

Example 5: Determine $\int \frac{x^7 - x + 3x^4}{x^5}\,dx$.

$$\int \frac{x^7 - x + 3x^4}{x^5}\,dx = \int\left(\frac{x^7}{x^5} - \frac{x}{x^5} + \frac{3x^4}{x^5}\right)dx$$

$$= \int \left(x^2 - x^{-4} + 3\left(\frac{1}{x}\right)\right)dx$$

$$= \frac{x^3}{3} - \frac{x^{-3}}{-3} + 3\ln|x| + c$$

$$= \boxed{\frac{x^3}{3} + \frac{1}{3x^3} + 3\ln|x| + c}$$

Example 6: Find $\int \frac{(\ln x)^4}{x}\,dx$.

$$\int \underbrace{(\ln x)^4}_{u^4}\underbrace{\left(\frac{1}{x}\right)dx}_{du} = \int u^4\,du$$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\,dx$

$du = \frac{1}{x}\,dx$

$$= \frac{u^5}{5} + C = \boxed{\frac{(\ln x)^5}{5} + C}$$

Example 7: Find $\int \frac{\ln(3x)}{x}\,dx$.

$$\int \underbrace{(\ln(3x))}_{u}\underbrace{\left(\frac{1}{x}\right)dx}_{du} = \int u\,du = \frac{u^2}{2} + C$$

$$= \boxed{\frac{(\ln(3x))^2}{2} + C}$$

Try $u = 3x$

$\frac{du}{dx} = 3$

$du = 3\,dx$

No...

Try $u = \ln(3x)$

$\frac{du}{dx} = \frac{1}{3x}(3) = \frac{1}{x}$

$du = \frac{1}{x}\,dx$

Example 8: Find $\int \frac{x}{x^2 - 8}\,dx$.

Example 9: Find $\int \frac{4x^2 - 5x - 12}{x^2 - 3}\,dx$.

Long Division

$$\begin{array}{r} 4 \\ x^2 - 3 \,\overline{)\, 4x^2 - 5x - 12} \\ -(4x^2 - 12) \\ \hline -5x + 0 \end{array}$$

$$\int \frac{4x^2 - 5x - 12}{x^2 - 3}\,dx = \int \left(4 + \frac{-5x}{x^2 - 3}\right)dx$$

$$= \int 4\,dx - \int \frac{5x}{x^2 - 3}\,dx$$

$$= \int 4\,dx - 5\int \underbrace{x\underbrace{\left(\frac{1}{x^2 - 3}\right)}_{1/u}dx}_{\frac{1}{2}du} = 4x - 5\left(\frac{1}{2}\right)\int \frac{1}{u}\,du$$

$u = x^2 - 3$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= 4x - \frac{5}{2}\ln|u| + C$$

$$= \boxed{4x - \frac{5}{2}\ln|x^2 - 3| + C}$$

Not done during class

Example 10: Find $\int \frac{4x^2-7x+1}{2x-3}\,dx$.

To long division:

$$\begin{array}{r} 2x \quad -\frac{1}{2} \\ 2x-3\,\overline{)\,4x^2-7x+1} \\ -(4x^2-6x) \\ \hline -x+1 \\ -\left(-x+\frac{3}{2}\right) \\ \hline -\frac{1}{2} \end{array}$$

$$\int \frac{4x^2-7x+1}{2x-3}\,dx = \int \left(2x-\frac{1}{2}+\frac{-\frac{1}{2}}{2x-3}\right)dx$$

$$= \int 2x\,dx - \int \frac{1}{2}\,dx - \frac{1}{2}\int \underbrace{\frac{1}{2x-3}}_{\frac{1}{u}}\,\underbrace{dx}_{\frac{1}{2}du}$$

$$= \frac{2x^2}{2} - \frac{1}{2}x - \frac{1}{2}\int \frac{1}{u}\left(\frac{1}{2}\right)du$$

$$= x^2 - \frac{1}{2}x - \frac{1}{4}\int \frac{1}{u}\,du$$

$$= x^2 - \frac{1}{2}x - \frac{1}{4}\ln|u| + c$$

$$= \boxed{x^2 - \frac{1}{2}x - \frac{1}{4}\ln|2x-3| + C}$$

$u = 2x-3$

$\frac{du}{dx} = 2$

$du = 2\,dx$

$\frac{1}{2}du = dx$

Integrating the remaining trigonometric functions:

Example 11: Determine $\int \tan x\,dx$.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \sin x\left(\frac{1}{\cos x}\right)dx$$

$\frac{1}{u}$ $\quad -du$

$$= -\int \frac{1}{u}\,du = -\ln|u| + c = \boxed{-\ln|\cos x| + c}$$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

Example 12: Determine $\int \cot x\,dx$.

Should get $\ln|\sin x| + c$

Example 13: Determine $\int \sec x \, dx$.

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

$u = \sec x + \tan x$

$\frac{du}{dx} = \sec x \tan x + \sec^2 x$

$du = (\sec x \tan x + \sec^2 x)\,dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$= \int \underbrace{\frac{1}{\sec x + \tan x}}_{\frac{1}{u}} \underbrace{(\sec^2 x + \sec x \tan x)\,dx}_{du}$$

$$= \int \frac{1}{u} \, du = \ln|u| + c = \boxed{\ln|\sec x + \tan x| + c}$$

Example 14: Determine $\int \csc x \, dx$.

Same process: multiply by $\frac{\csc x + \cot x}{\csc x + \cot x}$ or $\frac{\csc x - \cot x}{\csc x - \cot x}$

end up with

$-\ln|\csc x + \cot x| + c$

or $\ln|\csc x - \cot x| + c$

Homework Qs - Mon 4/20

4.5 #41

$$\int \frac{\csc^2 x}{\cot^3 x}\,dx = \int \frac{\frac{1}{\sin^2 x}}{\frac{\cos^3 x}{\sin^3 x}}\,dx$$

$$= \int \frac{1}{\sin^2 x}\cdot\frac{\sin^3 x}{\cos^3 x}\,dx = \int \frac{\sin x}{\cos^3 x}\,dx$$

much less scary now!

$u = \cos x$

$\frac{du}{dx} = -\sin x$

4.5 #51

$$\int \frac{x^2-1}{\sqrt{2x-1}}\,dx$$

$u = 2x-1$

$\frac{du}{dx} = 2$

$du = 2dx$

$\frac{1}{2}du = dx$

$u+1 = 2x$

$\frac{u+1}{2} = x$

$\frac{1}{2}u + \frac{1}{2} = x$

$$= \int (x^2-1)\underbrace{(2x-1)^{-1/2}}_{u^{-1/2}}\underbrace{dx}_{\frac{1}{2}du} = \frac{1}{2}\int (x^2-1)u^{-1/2}\,du$$

$$= \frac{1}{2}\int \left(x^2u^{-1/2} - u^{-1/2}\right)du$$

$$= \frac{1}{2}\int x^2u^{-1/2}\,du - \frac{1}{2}\int u^{-1/2}\,du$$

$$= \frac{1}{2}\int \left(\frac{u+1}{2}\right)^2 u^{-1/2}\,du - \frac{1}{\cancel{2}}\cdot\frac{u^{1/2}}{\cancel{1/2}} + C$$

$$= \frac{1}{2}\int \frac{u^2+2u+1}{4}u^{-1/2}\,du - u^{1/2} + C$$

$$= \frac{1}{8}\int (u^2+2u+1)u^{-1/2}\,du - u^{1/2} + C$$

$$= \frac{1}{8}\int \left(u^{3/2} + 2u^{1/2} + u^{-1/2}\right)du - u^{1/2} + C$$

$$= \frac{1}{8}\left(\frac{u^{5/2}}{5/2} + \frac{2u^{3/2}}{3/2} + \frac{u^{1/2}}{1/2}\right) - u^{1/2} + C$$

$$= \frac{1}{8}\left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} + 2u^{1/2}\right) - u^{1/2} + C$$

$$= \frac{1}{20}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + \frac{1}{4}(2x-1)^{1/2} - (2x-1)^{1/2} + C$$

$$= (2x-1)^{1/2}\left[\frac{1}{20}(2x-1)^2 + \frac{1}{6}(2x-1) - \frac{3}{4}\right]$$

5.1 #81

$y = x \ln x$

$y' = \frac{dy}{dx} = x\frac{d}{dx}(\ln x) + (\ln x)\frac{d}{dx}(x)$

$= x\left(\frac{1}{x}\right) + (\ln x)(1)$

$= 1 + \ln x$

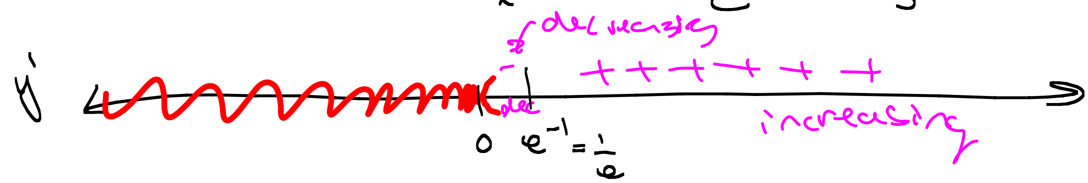
Set $y' = 0$: $0 = 1 + \ln x$

$-1 = \ln x$

$\ln x = -1$

$e^{\ln x} = e^{-1}$

$x = e^{-1} = \frac{1}{e} \approx \frac{1}{3}$ critical number



$(-\infty, \frac{1}{e})$ test number $x = e^{-2} = \frac{1}{e^2}$

Note: equation is only defined for $(0, \infty)$.

$y' = 1 + \ln x$

$y'\big|_{e^{-2}} = 1 + \ln e^{-2} = 1 - 2 = -1$

$(\frac{1}{e}, \infty)$: Test number $x = e \Rightarrow$

$y'\big|_{x=e} = 1 + \ln e = 1 + 1 = 2 \quad (+)$

If I chose test number $x = 1$, then

$y'\big|_{x=1} = 1 + \ln(1) = 1 + 0 = 1 \quad (+)$

$y' = 1 + \ln x$

$y'' = \frac{1}{x}$

where is y'' undefined? at $x = 0$

where is $y'' = 0$? It's never 0.

So no inflection pts.