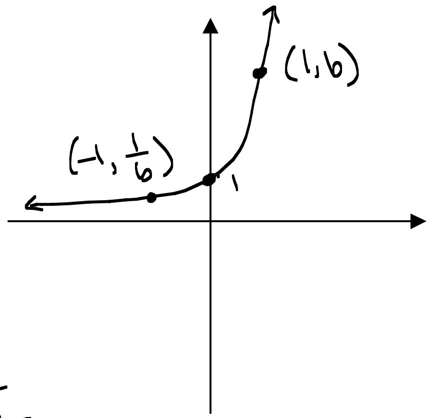
5.4: Exponential Functions: Differentiation and Integration

Short Review:

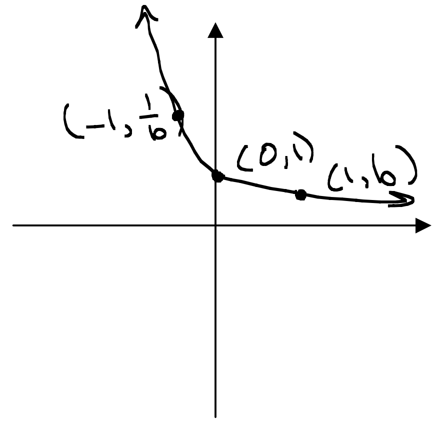
An *exponential* function takes the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

For any exponential function $f(x) = b^x$, the graph looks like one of the following.

x	$y = b^x$
0	$b^0 = 1$
1	$b^1 = b$
−1	$b^{-1} = \frac{1}{b}$



$b > 1$



$0 < b < 1$

Notice:

- Domain is $(-\infty, \infty)$.
- Range is $(0, \infty)$.
- Horizontal asymptote is $y = 0$.
- Always passes through the points $(0,1), (1,b), (-1, \frac{1}{b})$

The natural exponential function:

The number e can be defined in several ways.

One definition of the number e:

e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

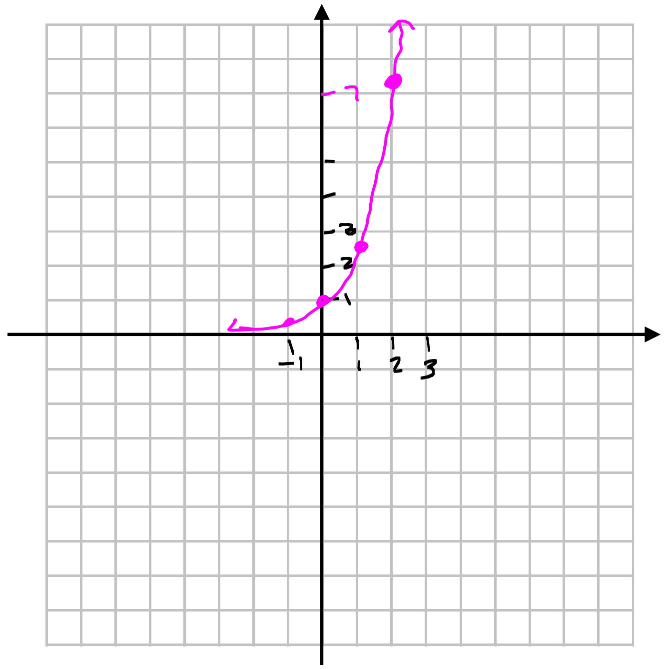
$f(x) = e^x$

$$\boxed{e \approx 2.718281828459}$$

The slope of the tangent line at the point $(0,1)$ is equal to 1.

The graph of $f(x) = e^x$:

x	$y = e^x$
1	$e^1 \approx 2.7$
0	$e^0 = 1$
-1	$e^{-1} = \frac{1}{e} \approx \frac{1}{3}$
2	$e^2 \approx 7.38$



Another definition of the number e:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \quad \text{or, equivalently,} \quad e = \lim_{x\to 0}(1+x)^{1/x}$$

Note:

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Let $h = \frac{1}{x}$

as $x \to \infty$, $h \to 0$

This gives us

$\lim_{h\to 0}(1+h)^{1/h}$

rewrite with x

$h = \frac{1}{x} \Rightarrow hx = 1 \Rightarrow x = \frac{1}{h}$

Derivatives of exponential functions:

$$\frac{d}{dx}\left(e^x\right) = e^x$$

Note: $f'(0) = e^0 = 1 \Leftarrow \begin{cases} f(x) = e^x \\ f'(x) = e^x \end{cases}$

__Example 1:__ Find the derivative of $f(x) = -7e^x$.

$f'(x) = -7e^x$

Ex $3\frac{1}{2}$: $\frac{d}{dx}(\cos(-7x)) = -\sin(-7x)\frac{d}{dx}(-7x)$

$= -(\sin(-7x))(-7) = \boxed{7\sin(-7x)}$

Example 2: Find the derivative of $f(x) = 5\sqrt{e^x + 7}$.

$f(x) = 5(e^x+7)^{1/2}$

$f'(x) = 5\left(\frac{1}{2}\right)(e^x+7)^{-1/2}\frac{d}{dx}(e^x+7) = \frac{5}{2}(e^x+7)^{-1/2}(e^x) = \boxed{\frac{5e^x}{2\sqrt{e^x+7}}}$

Example 3: Find the derivative of $f(x) = e^x \sin x$.

$f'(x) = e^x\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(e^x)$

$= e^x \cos x + (\sin x)(e^x) = \boxed{e^x\cos x + e^x \sin x}$

$= \boxed{e^x(\cos x + \sin x)}$

Example 4: Find the derivative of $g(x) = e^{-7x} + 2x^3 - 4e$.

$g'(x) = e^{-7x}\frac{d}{dx}(-7x) + 6x^2 + 0 = e^{-7x}(-7) + 6x^2 = \boxed{-7e^{-7x} + 6x^2}$

Example 5: Find the derivative of $y = e^{x^2+4x}$.

$\frac{dy}{dx} = e^{x^2+4x}\frac{d}{dx}(x^2+4x) = \boxed{e^{x^2+4x}(2x+4)}$

Example 6: Find the derivative of $f(x) = \cos(e^x - x)$.

$f'(x) = -\sin(e^x - x)\frac{d}{dx}(e^x - x)$

$= -[\sin(e^x - x)](e^x - 1) = -(e^x-1)\sin(e^x-x)$

$= \boxed{(1-e^x)\sin(e^x-x)}$

Example 7: Find the equation of the tangent line to the graph of $f(x) = (e^x + 2)^2$ at the point $(0,9)$.

$f'(x) = 2(e^x+2)\frac{d}{dx}(e^x+2) = 2(e^x+2)(e^x) = 2e^x(e^x+2)$

$m = f'(0) = 2e^0(e^0+2) = 2(1)(1+2) = 2(3) = 6$

Verify: $f(0) = (e^0+2)^2 = (1+2)^2 = 3^2 = 9$ ✓

$y - y_1 = m(x - x_1)$

$y - 9 = 6(x - 0)$

$y - 9 = 6x$

$\boxed{y = 6x + 9}$

Integration of exponential functions:

Check:
$\frac{d}{dx}(e^x + c)$
$= e^x$ ✓ ok

$$\boxed{\int e^x dx = e^x + c}$$

Example 8: Determine $\int (x^2 - 5e^x)\,dx$

$$\int x^2\,dx - 5\int e^x\,dx$$
$$= \boxed{\frac{x^3}{3} - 5e^x + c}$$

Example 9: Find $\int e^{5t}\,dt$.

$u = 5t$
$\frac{du}{dt} = 5$
$du = 5\,dt$
$\frac{1}{5}du = dt$

$$\int \underbrace{e^{5t}}_{e^u}\,\underbrace{dt}_{\frac{1}{5}du} = \frac{1}{5}\int e^u\,du$$
$$= \frac{1}{5}e^u + c$$
$$= \boxed{\frac{1}{5}e^{5t} + c}$$

Check:
$$\frac{d}{dt}\left(\frac{1}{5}e^{5t}\right) = \frac{1}{5}e^{5t}\frac{d}{dt}(5t)$$
$$= \frac{1}{\cancel{5}}e^{5t}(\cancel{5})$$
$$= e^{5t}$$ ✓

not done during class

Example 10: Find $\int_1^3 e^{2x-3}\,dx$.

$u = 2x - 3$
$\frac{du}{dx} = 2$
$du = 2dx$
$\frac{1}{2}du = dx$
$x = 1 \Rightarrow u = 2(1) - 3 = 2 - 3 = -1$
$x = 3 \Rightarrow u = 2(3) - 3 = 6 - 3 = 3$

$$\int_1^3 e^{2x-3}\,dx = \frac{1}{2}\int_{u=-1}^{u=3} e^u\,du$$
$$= \frac{1}{2}e^u\Big|_{u=-1}^{u=3} = \boxed{\frac{1}{2}e^3 - \frac{1}{2}e^{-1}}$$

OR (without changing limits of integration)

$$\int_1^3 e^{2x-3}\,dx = \frac{1}{2}\int_{x=1}^{x=3} e^u\,du = \frac{1}{2}e^u\Big|_{x=1}^{x=3}$$
$$= \frac{1}{2}e^{2x-3}\Big|_{x=1}^{x=3} = \frac{1}{2}e^{2(3)-3} - \frac{1}{2}e^{2(1)-3}$$
$$= \frac{1}{2}e^{6-3} - \frac{1}{2}e^{2-3} = \boxed{\frac{1}{2}e^3 - \frac{1}{2}e^{-1}}$$

Example 11: Find $\int te^{t^2}\,dt$.

$u = t^2$

$\frac{du}{dt} = 2t$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$$\int \underbrace{t\,\underbrace{e^{t^2}}_{e^u}\,dt}_{\frac{1}{2}du} = \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + c$$

$$= \boxed{\frac{1}{2}e^{t^2} + c}$$

Example 12: Determine $\int \frac{e^x}{\sqrt[3]{e^x+1}}\,dx$.

$u = e^x + 1$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

$$\int \frac{e^x}{\sqrt[3]{e^x+1}}\,dx = \int \underbrace{e^x\,\underbrace{(e^x+1)^{-1/3}}_{u^{-1/3}}\,dx}_{du}$$

$$= \int u^{-1/3}\,du$$

$$= \frac{u^{2/3}}{2/3} + c = \frac{3}{2}u^{2/3} + c = \boxed{\frac{3}{2}(e^x+1)^{2/3} + c}$$

Example 13: Determine $\int \frac{e^x - e^{-x}}{e^{3x}}\,dx$

$$\int \frac{e^x - e^{-x}}{e^{3x}}\,dx = \int \left(\frac{e^x}{e^{3x}} - \frac{e^{-x}}{e^{3x}}\right)dx$$

$$= \int \left(e^{x-3x} - e^{-x-3x}\right)dx = \int \left(e^{-2x} - e^{-4x}\right)dx$$

$$= \int e^{-2x}\,dx - \int e^{-4x}\,dx$$

$u = -2x$

$\frac{du}{dx} = -2$

$du = -2\,dx$

$-\frac{1}{2}du = dx$

$$= -\frac{1}{2}\int e^u\,du - \left(-\frac{1}{4}\right)\int e^{\hat{u}}\,d\hat{u}$$

$$= -\frac{1}{2}e^u + \frac{1}{4}e^{\hat{u}} + c$$

$$= \boxed{-\frac{1}{2}e^{-2x} + \frac{1}{4}e^{-4x} + c}$$

$\hat{u} = -4x$

$\frac{d\hat{u}}{dx} = -4$

$d\hat{u} = -4\,dx$

$-\frac{1}{4}d\hat{u} = dx$