5.6: Inverse Trigonometric Functions – Differentiation

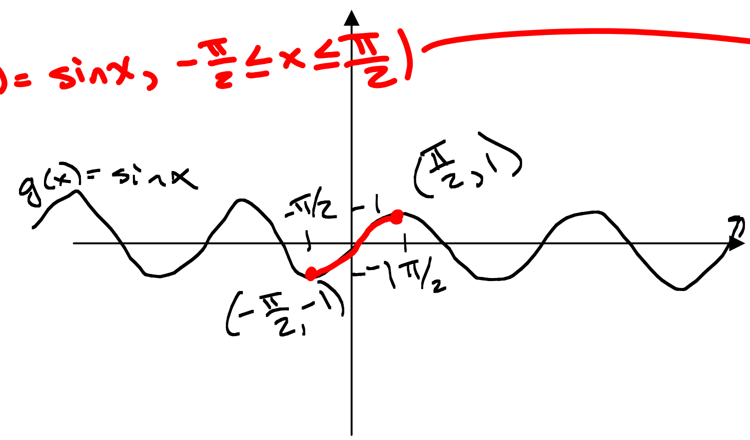
Because none of the trigonometric functions are one-to-one, none of them have an inverse function. To overcome this problem, the domain of each function is restricted so as to produce a one-to-one function.

same as arcsinx

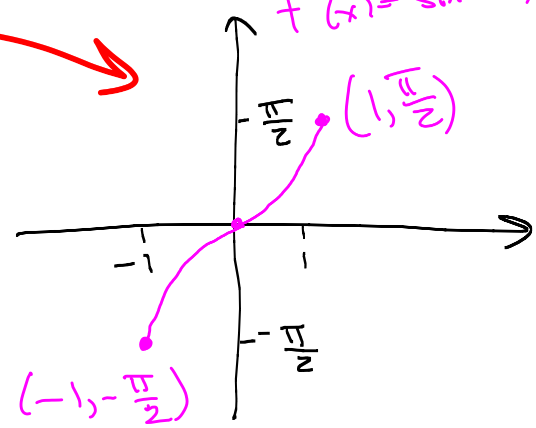
Inverse sine function:

$$y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \ \text{ and } \ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$f(x) = \sin x, \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



$f^{-1}(x) = \sin^{-1}(x)$



For $f^{-1}(x) = \sin^{-1}(x)$

Domain: $[-1, 1]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

<u>Properties of the inverse sine function</u>:

$$\sin^{-1}(\sin x) = x \ \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \ \text{ for } -1 \le x \le 1$$

<u>Example 1:</u> Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\sin^{-1}\left(-\frac{1}{2}\right)$.

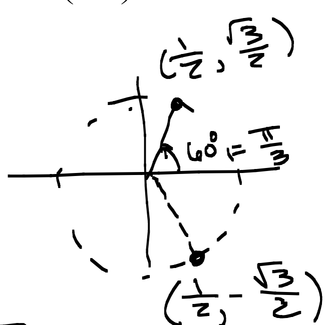
$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

This means that

$\sin\theta = \frac{\sqrt{3}}{2}$

and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi}{3}}$

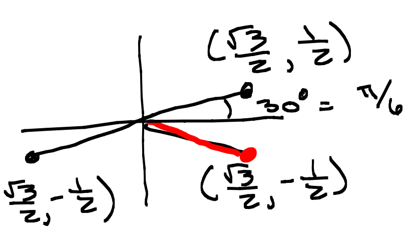


2nd part

Let $\theta = \sin^{-1}\left(-\frac{1}{2}\right)$

This means that $\sin\theta = -\frac{1}{2}$

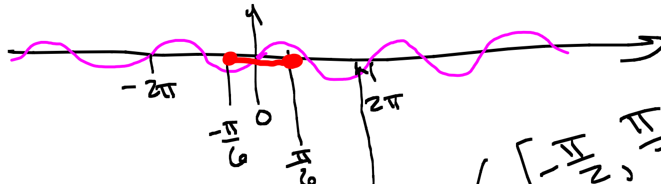
and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



$\theta = \sin^{-1}\left(-\frac{1}{2}\right) = \boxed{-\frac{\pi}{6}}$

see next page

Why not $\frac{11\pi}{6}$?



$\frac{11\pi}{6} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

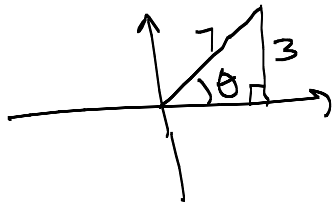
Example 2: Evaluate $\cot\left(\sin^{-1}\frac{3}{7}\right)$

Let $\theta = \sin^{-1}\left(\frac{3}{7}\right)$

Then $\sin\theta = \frac{3}{7}$

and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\theta > 0$, so θ must be in Quadrant I



Find the missing side:

$a^2 + 3^2 = 7^2$

$a^2 + 9 = 49$

$a^2 = 40$

$a = \pm\sqrt{40} = \pm 2\sqrt{10}$

Quadrant I, so (+)

$\cot\left(\sin^{-1}\frac{3}{7}\right) = \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$

$= \boxed{\frac{2\sqrt{10}}{3}}$

Ex $2\frac{1}{2}$: Evaluate $\sin^{-1}(-1.5)$

suppose $\theta = \sin^{-1}(-1.5)$

Then $\sin\theta = -1.5$, which is impossible

$\boxed{\sin^{-1}(-1.5) \text{ is undefined.}}$

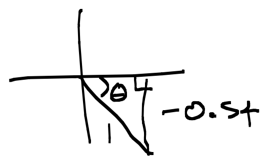
Example 3: Evaluate $\sin\left(\sin^{-1}(-0.54)\right)$.

$\theta = \sin^{-1}(-0.54)$

$\sin\theta = -0.54$ no problem

and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Quadrant IV



$\sin\theta$

$= \sin\left(\sin^{-1}(-0.54)\right)$

$= \boxed{-0.54}$

Example 4: Evaluate $\sin\left(\sin^{-1}2\right)$.

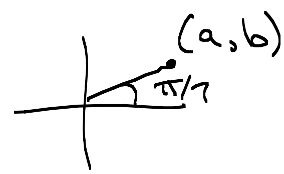
$\theta = \sin^{-1}2$

Then $\sin\theta = 2$ impossible!

So $\sin^{-1}2$ is undefined. Therefore, $\sin(\sin^{-1}2)$ is undefined.

Example 5: Evaluate $\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \boxed{\frac{\pi}{4}}$ because $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

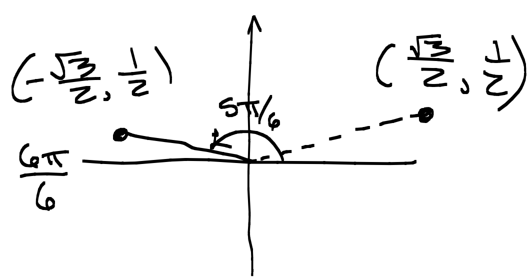
$\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\pi}{4}}$



Ex $5\frac{1}{2}$: $\sin^{-1}\left(\sin\frac{\pi}{7}\right) = \sin^{-1}(b) = \boxed{\frac{\pi}{7}}$

Example 6: Evaluate $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

$\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$



For $\theta = \sin^{-1}\left(\frac{1}{2}\right)$,

then $\sin\theta = \frac{1}{2}$

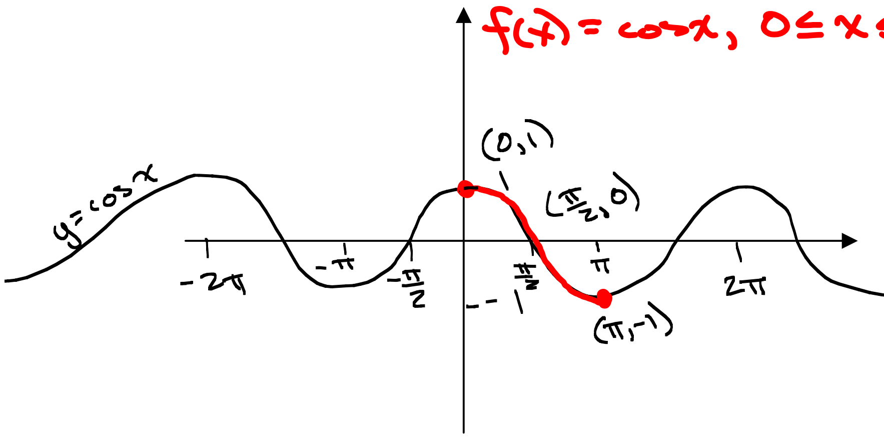
and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

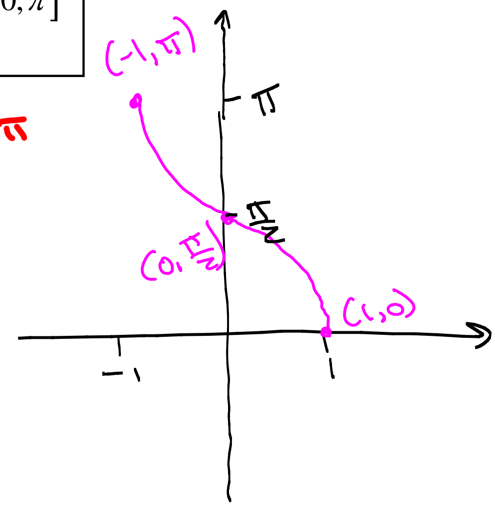
So $\theta = \frac{\pi}{6}$

Inverse cosine function: same as $y = \arccos x$

$$y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \ \text{ and } \ y \in [0, \pi]$$

$f^{-1}(x) = \cos^{-1} x$





Domain of $\cos^{-1}(x)$: $[-1, 1]$
Range of $\cos^{-1}(x)$: $[0, \pi]$

Properties of the inverse cosine function:

$$\cos^{-1}(\cos x) = x \ \text{ for } \ 0 \le x \le \pi$$
$$\cos(\cos^{-1} x) = x \ \text{ for } \ -1 \le x \le 1$$

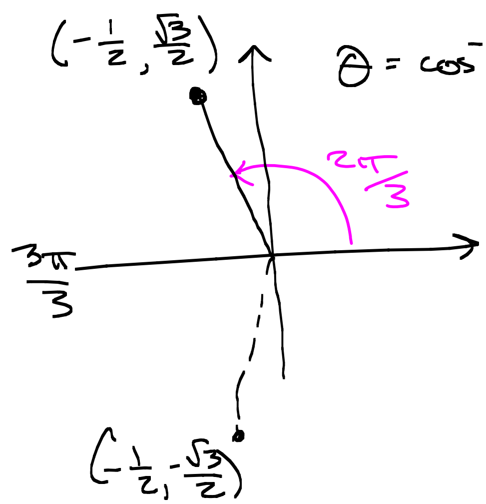
Example 7: Evaluate $\cos^{-1}\left(-\frac{1}{2}\right)$

Let $\theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$\cos\theta = -\frac{1}{2}$

and $\theta \in [0, \pi]$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3}}$



Example 8: Evaluate $\cos^{-1}\left(\cos\left(-\frac{5\pi}{6}\right)\right)$

$\cos^{-1}\left(\cos\left(-\frac{5\pi}{6}\right)\right)$

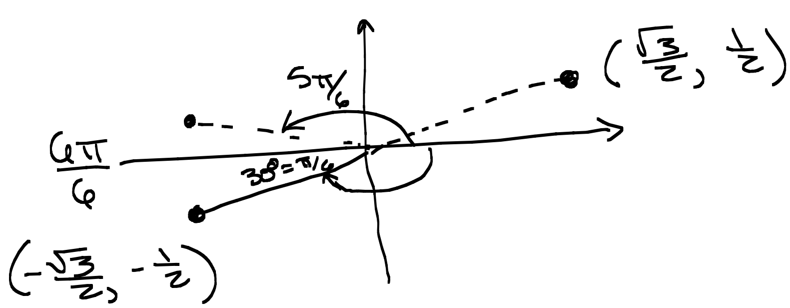
$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

For $\theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$,

we have $\cos\theta = -\frac{\sqrt{3}}{2}$

and $\theta \in [0, \pi]$

So $\theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$



Inverse tangent function:

$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

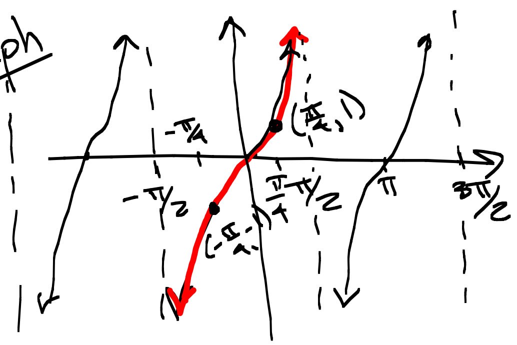
$y = \tan^{-1} x$ same as $y = \arctan x$

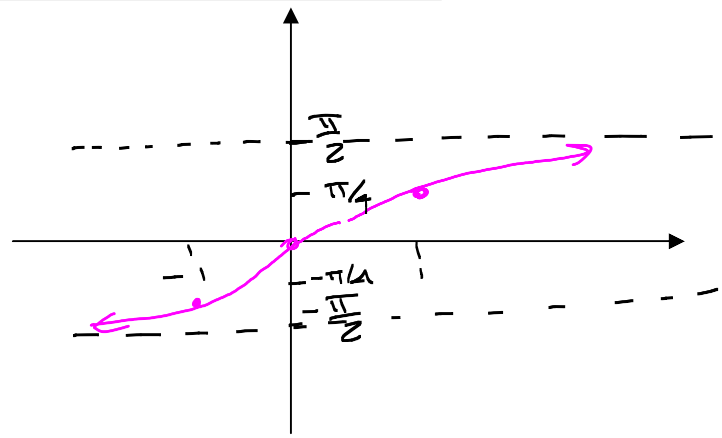
$$y = \tan^{-1} x \quad \text{if and only if} \quad \tan y = x \ \text{ and } \ y \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \ \text{ and } \ \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Graph of $y = \tan^{-1} x$ **:**

Know this graph





Inverse secant, cosecant, and cotangent functions:

These are not used as often, and are not defined consistently. Our book defines them as follows:

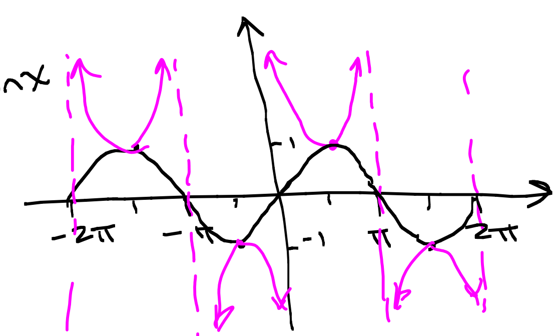
$$y = \cot^{-1} x \quad \text{if and only if} \quad \cot y = x \ \text{ and } \ y \in (0, \pi)$$

$$y = \csc^{-1} x \quad \text{if and only if} \quad \csc y = x \ \text{ and } \ y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$$

$$y = \sec^{-1} x \quad \text{if and only if} \quad \sec y = x \ \text{ and } \ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

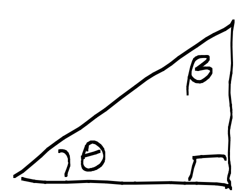
$y = \csc x$

1st graph $y = \sin x$



<u>An important identity:</u>

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$



Note: all angles must add up to 180°, so $\theta + \beta = \frac{\pi}{2}$

Differentiation of the inverse sine function:

Since $y = \sin x$ is continuous and differentiable, so is $y = \sin^{-1} x$.

We want to find its derivative.

$y = \sin^{-1} x$. Find $\frac{dy}{dx}$

$x = \sin y$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Implicit diff.:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$1 = (\cos y)\frac{dy}{dx}$$

$$\frac{1}{\cos y} = \frac{dy}{dx}$$

We want to write $\frac{1}{\cos y}$ in terms of x.

we know that $\cos^2 y + \sin^2 y = 1$ [Pythagorean identity]

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \pm\sqrt{1-\sin^2 y}$$

Because we know that $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we know $\cos y > 0$

So $\cos y = +\sqrt{1-\sin^2 y}$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$x = \sin y$, so $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\boxed{\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \quad , -1 < x < 1}$$

Example 9: Differentiate $f(x) = \sin^{-1}(2x-7)$.

$$f'(x) = \frac{1}{\sqrt{1-(2x-7)^2}} \frac{d}{dx}(2x-7)$$

$$= \frac{1}{\sqrt{1-(2x-7)^2}}(2) = \frac{2}{\sqrt{1-(4x^2-28x+49)}}$$

$$= \boxed{\frac{2}{\sqrt{-4x^2+28x-48}}}$$

Differentiation of the inverse cosine function:

$\frac{d}{dx}(\cos^{-1}x)$

$= -\frac{1}{\sqrt{1-x^2}}$

$$\boxed{\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}} \quad , -1 < x < 1}$$

tangent

Differentiation of the inverse ~~cosine~~ function:

$$\boxed{\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}}$$

Derivatives of other inverse trigonometric functions:

Know these 3

no need to memorize these 3.

$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left(\csc^{-1}x\right) = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$	$\frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}$

Example 10: Find the derivative of $f(x) = e^{\tan^{-1} 2x}$.

$f(x) = e^{\tan^{-1} 2x}$

$f'(x) = e^{\tan^{-1} 2x} \frac{d}{dx}(\tan^{-1} 2x) = e^{\tan^{-1} 2x}\left(\frac{1}{1+(2x)^2}\right)\frac{d}{dx}(2x)$

$= e^{\tan^{-1} 2x}\left(\frac{1}{1+4x^2}\right)(2) = \boxed{\frac{2e^{\tan^{-1} 2x}}{1+4x^2}}$

Example 11: Find the derivative of $y = \csc^{-1}(\tan x)$.

Look up formula: $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

$\frac{dy}{dx} = -\frac{1}{|\tan x|\sqrt{(\tan x)^2-1}} \frac{d}{dx}(\tan x) = -\frac{1}{|\tan x|\sqrt{\tan^2 x-1}} \sec^2 x$

$= \boxed{-\frac{\sec^2 x}{|\tan x|\sqrt{\tan^2 x-1}}}$

Example 12: Find the derivative of $f(x) = x^3 \arccos 2x$.

$f'(x) = x^3 \frac{d}{dx}(\arccos 2x) + \arccos 2x \frac{d}{dx}(x^3)$

$= x^3\left(-\frac{1}{\sqrt{1-(2x)^2}}\right)\frac{d}{dx}(2x) + (\arccos(2x))(3x^2)$

$= \boxed{-\frac{2x^3}{\sqrt{1-4x^2}} + 3x^2 \arccos(2x)}$

Example 13: Find the equation of the line tangent to the graph of $f(x) = \arctan x$ at the point where $x = -1$.

Find the ordered pair: $f(-1) = \arctan(-1)$

$= \tan^{-1}(-1) = -\frac{\pi}{4}$

(For $\theta = \tan^{-1}(-1)$, then $\tan\theta = -1$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

$f(x) = \arctan x$

$f'(x) = \frac{1}{1+x^2}$

$m = f'(-1) = \frac{1}{1+(-1)^2} = \frac{1}{1+1} = \frac{1}{2}$

$y - y_1 = m(x - x_1)$

$y - \left(-\frac{\pi}{4}\right) = \frac{1}{2}(x-(-1))$

$y + \frac{\pi}{4} = \frac{1}{2}(x+1)$

$\boxed{y = \frac{1}{2}x + \frac{1}{2} - \frac{\pi}{4}}$