5.7: Inverse Trigonometric Functions – Integration

Two important integration rules come from the inverse trigonometric differentiation rules.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$$

From 5.6

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{x^2+1}$$

Note: $\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\left(-\cos^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$

So, $\int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1}x + c$

Hmmm.... is this the same as $\sin^{-1}x + c$?

Recall: $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\sin^{-1}x = -\cos^{-1}x + \frac{\pi}{2}$

so $\sin^{-1}x + c = -\cos^{-1}x + \hat{c}$

Example 1: Find $\int \frac{1}{\sqrt{1-9x^2}}\,dx$.

$$\int \frac{1}{\sqrt{1-(3x)^2}}\,dx$$

want $\int \frac{1}{\sqrt{1-u^2}}\,du$

$u = 3x$

$\frac{du}{dx} = 3$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{3}\sin^{-1}(u) + c$$

$$= \boxed{\frac{1}{3}\sin^{-1}(3x) + C}$$

Example 2: Find $\int \frac{1}{x^2+7}\,dx$.

want $\int \frac{1}{u^2+1}\,du$

$$\int \frac{1}{x^2+7}\,dx$$

$$= \int \frac{1}{7\left(\frac{x^2}{7}+1\right)}\,dx = \frac{1}{7}\int \frac{1}{\frac{x^2}{7}+1}\,dx = \frac{1}{7}\int \frac{1}{\left(\frac{x}{\sqrt{7}}\right)^2+1}\,\underbrace{dx}_{\sqrt{7}\,du}$$

$u = \frac{x}{\sqrt{7}} = \frac{1}{\sqrt{7}}x$

$\frac{du}{dx} = \frac{1}{\sqrt{7}}$

$du = \frac{1}{\sqrt{7}}\,dx$

$\sqrt{7}\,du = dx$

$$= (\sqrt{7})\frac{1}{7}\int \frac{1}{u^2+1}\,du = \frac{\sqrt{7}}{7}\tan^{-1}(u) + c$$

$$= \boxed{\frac{\sqrt{7}}{7}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + C}$$

More general forms of these integration rules are

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C.$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$$

<u>Ex:</u> $\int \frac{1}{x^2+13}\,dx$

Use $a=\sqrt{13}$ $\quad = \int \frac{1}{x^2+(\sqrt{13})^2}\,dx$

$$= \boxed{\frac{1}{\sqrt{13}}\tan^{-1}\left(\frac{x}{\sqrt{13}}\right)+C}$$

Example 3: Find $\int \frac{4x}{x^4+25}\,dx$.

$$4\int \frac{x}{x^4+25}\,dx = 4\int \frac{x\,dx}{(x^2)^2+25}$$

$\frac{1}{2}du$

$u = x^2$
$\frac{du}{dx} = 2x$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$= 4\left(\frac{1}{2}\right)\int \frac{1}{u^2+25}\,du = 2\int \frac{1}{u^2+5^2}\,du$$

$$= 2\left(\frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right)\right)+C$$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$ we're using $a=5$

$$= \frac{2}{5}\tan^{-1}\left(\frac{u}{5}\right)+C = \boxed{\frac{2}{5}\tan^{-1}\left(\frac{x^2}{5}\right)+C}$$

<u>Check:</u> $\frac{d}{dx}\left(\frac{2}{5}\tan^{-1}\left(\frac{x^2}{5}\right)\right)$

$$= \frac{2}{5}\cdot\frac{1}{1+\left(\frac{x^2}{5}\right)^2}\,\frac{d}{dx}\left(\frac{x^2}{5}\right)$$

$$= \frac{2}{5}\cdot\frac{1}{1+\frac{x^4}{25}}\cdot\frac{1}{5}\left(\frac{2x}{1}\right)$$

$$= \frac{4x}{25\left(1+\frac{x^4}{25}\right)} = \frac{4x}{25+x^4} \checkmark$$

Another antiderivative:

$$\boxed{\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x|+C}$$

Example 4: Find $\int \frac{1}{x\sqrt{x^2-4}}\,dx$.

Example 5: Find $\int \frac{1}{4x^2-12x+17}dx$.

Complete the square:

$$4x^2-12x+17$$
$$=(4x^2-12x)+17$$
$$=4(x^2-3x)+17$$
$$=4\left(x^2-3x+\frac{9}{4}\right)+17-9 \qquad -4\left(\frac{9}{4}\right)$$
$$\left(\frac{-3}{2}\right)^2=\frac{9}{4}$$
$$=4\left(x-\frac{3}{2}\right)^2+8$$

$$=\int \frac{1}{4(x-\frac{3}{2})^2+8}\,dx$$

$$=\frac{1}{4}\int \frac{1}{(x-\frac{3}{2})^2+2}\,dx$$

$u=x-\frac{3}{2}$

$\frac{du}{dx}=1$

$du=dx$

$$=\frac{1}{4}\int \frac{1}{u^2+2}\,du$$

use $\int \frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$ with $a=\sqrt{2}$

$$=\frac{1}{4}\int \frac{1}{u^2+(\sqrt{2})^2}\,du$$

$$=\frac{1}{4}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\right]+c=\boxed{\frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{3}{2}}{\sqrt{2}}\right)+c}$$

Example 6: Find $\int \frac{x-7}{\sqrt{5-4x^2}}dx$.

$$\int \frac{x-7}{\sqrt{5-4x^2}}dx=\int \frac{x}{\sqrt{5-4x^2}}dx-\int \frac{7}{\sqrt{5-4x^2}}dx$$

For I_1:

$u=5-4x^2$

$du=-8x\,dx$

$-\frac{1}{8}du=x\,dx$

$$I_1=\int \frac{x}{\sqrt{5-4x^2}}dx=-\frac{1}{8}\int \frac{1}{\sqrt{u}}du=-\frac{1}{8}\int u^{-1/2}du$$

$$=-\frac{1}{8}\cdot\frac{u^{1/2}}{1/2}+c=-\frac{1}{8}\cdot\frac{2}{1}u^{1/2}+c=-\frac{1}{4}u^{1/2}+c$$

$$=-\frac{1}{4}\sqrt{5-4x^2}+c$$

For I_2:

$u=2x$

$\frac{du}{dx}=2$

$du=2dx$

$\frac{1}{2}du=dx$

$$I_2=\int \frac{7}{\sqrt{5-4x^2}}dx=7\int \frac{1}{\sqrt{5-(2x)^2}}dx$$

$$=7\left(\frac{1}{2}\right)\int \frac{1}{\sqrt{5-u^2}}du=\frac{7}{2}\int \frac{1}{\sqrt{(\sqrt{5})^2-u^2}}du$$

$$=\frac{7}{2}\sin^{-1}\left(\frac{u}{\sqrt{5}}\right)+c$$

$$=\frac{7}{2}\sin^{-1}\left(\frac{2x}{\sqrt{5}}\right)+c$$

Use $\int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+c$ with $a=\sqrt{5}$

So original integral is

$$\int \frac{x-7}{\sqrt{5-4x^2}}dx=I_1-I_2$$

$$=\boxed{-\frac{1}{4}\sqrt{5-4x^2}-\frac{7}{2}\sin^{-1}\left(\frac{2x}{\sqrt{5}}\right)+c}$$

Homework Qs

5.7 #17: $\int \frac{x-3}{x^2+1}\,dx$ Hint: write as 2 pieces.

$$= \int \frac{x}{x^2+1}\,dx - \int \frac{3}{x^2+1}\,dx$$