5.8: Hyperbolic Functions

The unit circle $x^2 + y^2 = 1$ is used in trigonometry to define the trigonometric functions. Often they are called "circular functions." In a similar manner, the unit hyperbola $x^2 - y^2 = 1$ can be used to define functions that have many similar characteristics as the circular trigonometric functions. Such functions are called "hyperbolic functions." These hyperbolic functions can also be defined in terms of the natural exponential function.

$y^2 - x^2 = 1$

sinh: hyperbolic sine
cosh: hyperbolic cosine
etc.

$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Recall: odd and even functions

We can also think of decomposing e^x into odd and even parts:

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

The odd part is $\dfrac{e^x - e^{-x}}{2} = \sinh x$ and the even part is $\dfrac{e^x + e^{-x}}{2} = \cosh x$.

Example 1: Evaluate (a) $\cosh 2$ (b) $\tanh 0$ (c) $\sinh 1$

(a) $\cosh 2 = \dfrac{e^2 + e^{-2}}{2}$

(b) $\tanh(0) = \dfrac{e^0 - e^{-0}}{e^0 + e^{-0}} = \dfrac{1-1}{1+1} = \dfrac{0}{2} = \boxed{0}$

(c) $\sinh(1) = \dfrac{e^1 - e^{-1}}{2} = \dfrac{e - \frac{1}{e}}{2} = \dfrac{e - \frac{1}{e}}{2}\left(\dfrac{e}{e}\right)$

$= \boxed{\dfrac{e^2 - 1}{2e}}$

Hyperbolic functions have many uses in science and engineering applications.

Hyperbolic identities:

Several identities involving hyperbolic functions are similar to the usual trigonometric identities, but there are a few important differences.

$\sinh(-x) = -\sinh x$	$\cosh(-x) = \cosh x$
$\cosh^2 x - \sinh^2 x = 1$	$1 - \tanh^2 x = \operatorname{sech}^2 x$
$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$	
$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$	
$\sinh 2t = 2\sinh t \cosh t$	
$\cosh 2t = \cosh^2 t + \sinh^2 t$	

Derivatives of hyperbolic functions:

$\frac{d}{dx}(\sinh x) = \cosh x$	$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
$\frac{d}{dx}(\cosh x) = \sinh x$	$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$	$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

<u>**Example 2:**</u> Find $\frac{dy}{dx}$ for $y = \tanh\sqrt{3x^2-5}$.

$$y = \tanh(3x^2-5)^{1/2}$$

$$\frac{dy}{dx} = \left[\operatorname{sech}^2(3x^2-5)^{1/2}\right]\frac{d}{dx}(3x^2-5)^{1/2}$$

$$= \left[\operatorname{sech}^2(3x^2-5)^{1/2}\right]\left(\tfrac{1}{2}\right)(3x^2-5)^{-1/2}(6x)$$

$$= \boxed{\frac{3x\operatorname{sech}^2(3x^2-5)^{1/2}}{\sqrt{3x^2-5}}}$$

Inverse hyperbolic functions:

The functions sinh and tanh are one-to-one functions, which means they have inverse functions. The function cosh, however, is not one-to-one. But, restricting the domain of cosh to $[0,\infty)$ defines a one-to-one function.

$y = \sinh^{-1} x$ if and only if $\sinh y = x$
$y = \cosh^{-1} x$ if and only if $\cosh y = x$ and $y \ge 0$
$y = \tanh^{-1} x$ if and only if $\tanh y = x$

Because the hyperbolic functions are defined in terms of the natural exponential function, it should come as no surprise that the inverse hyperbolic functions are defined in terms of the natural logarithmic function.

$\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right),\ x \in \mathbb{R}$
$\cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right),\ x \geq 1$
$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ -1 < x < 1$

Derivatives of Inverse Hyperbolic Functions

$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}$	$\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$
$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$	$\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1-x^2}}$
$\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$	$\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Note: Even though the derivativse of $\tanh^{-1} x$ and $\coth^{-1} x$ look identical, they are really different because these inverse functions are defined on different domains: $\lvert x\rvert < 1$ for $\tanh^{-1} x$, and $\lvert x\rvert > 1$ for $\coth^{-1} x$.

Example 3: Find $\frac{d}{dx}\left[\sinh^{-1}(\tan x)\right]$.

$$\frac{d}{dx}\left[\sinh^{-1}(\tan x)\right] = \frac{1}{\sqrt{1+\tan^2 x}}\,\frac{d}{dx}(\tan x)$$

$$= \frac{1}{\sqrt{1+\tan^2 x}}\,(\sec^2 x)$$

$$= \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{\lvert\sec x\rvert} = \boxed{\lvert\sec x\rvert}$$

must be positive, as numerator and denominator are both positive.

So the simplified version has to be positive also.

Note: $\sqrt{x^2} = \lvert x\rvert$

$\sqrt{x^2} = x$ is only true when $x \geq 0$

Example 4: $\int \frac{3}{\sqrt{x^2-1}}\,dx$

We know that $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$.

So, $\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + c$

Therefore,

$$\int \frac{3}{\sqrt{x^2-1}}\,dx = 3\int \frac{1}{\sqrt{x^2-1}}\,dx = \boxed{3\cosh^{-1} x + c}$$

Example 5: Evaluate $\int \frac{4dx}{\sqrt{9x^2-1}}$.

$$\int \frac{4dx}{\sqrt{9x^2-1}} = 4\int \frac{1}{\sqrt{9x^2-1}}\,dx = 4\int \frac{1}{\sqrt{(3x)^2-1}}\,dx$$

$$= 4\left(\frac{1}{3}\right)\int \frac{1}{\sqrt{u^2-1}}\,du$$

$u = 3x$

$\frac{du}{dx} = 3$

$du = 3dx$

$\frac{1}{3}du = dx$

$$= \frac{4}{3}\cosh^{-1}(u) + c$$

$$= \boxed{\frac{4}{3}\cosh^{-1}(3x) + c}$$

Example 6: Evaluate $\int \coth x\,dx$

$$\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx$$

$u = \sinh x$

$\frac{du}{dx} = \cosh x$

$du = \cosh x\,dx$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + c = \boxed{\ln|\sinh x| + c}$$