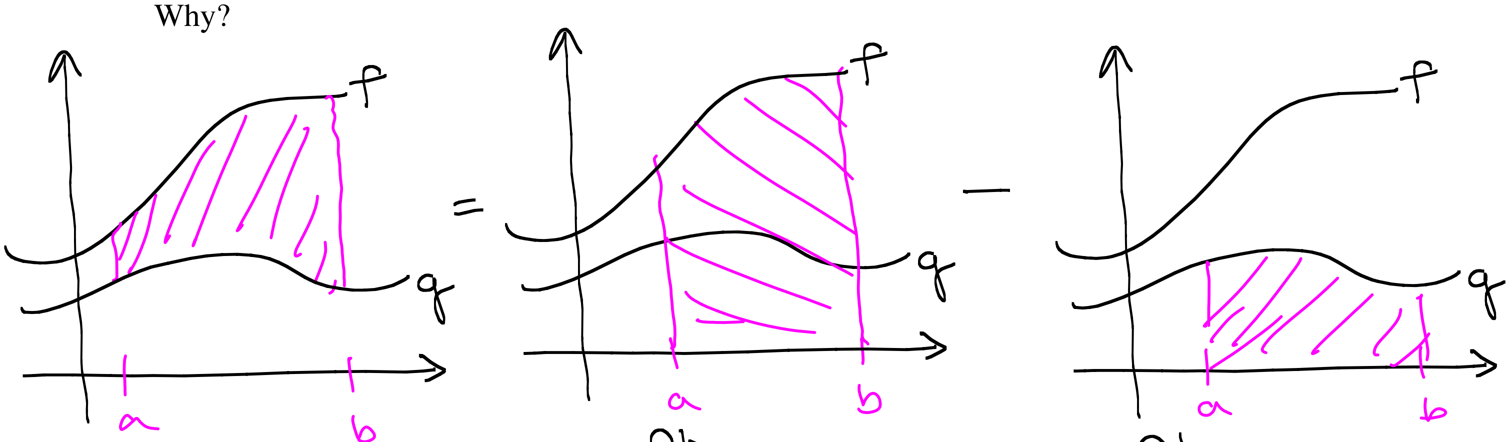
7.1: Area of a Region Between Two Curves

Because the definite integral represents the "net" area under a curve, we can use integration to find the area between curves.

If f and g are continuous and $f(x) \geq g(x)$ on $[a,b]$, then the area between $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b [f(x) - g(x)]dx$$

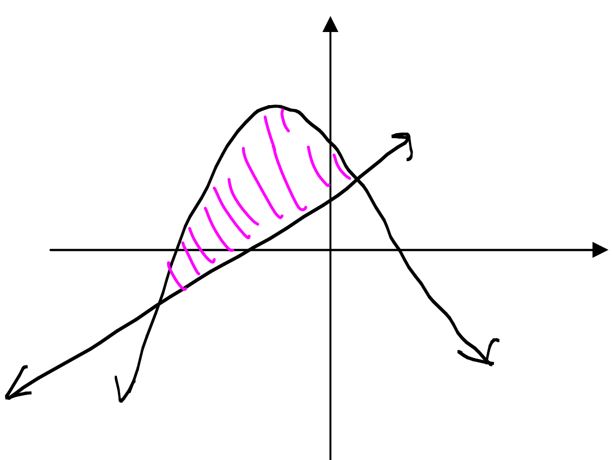
Why?



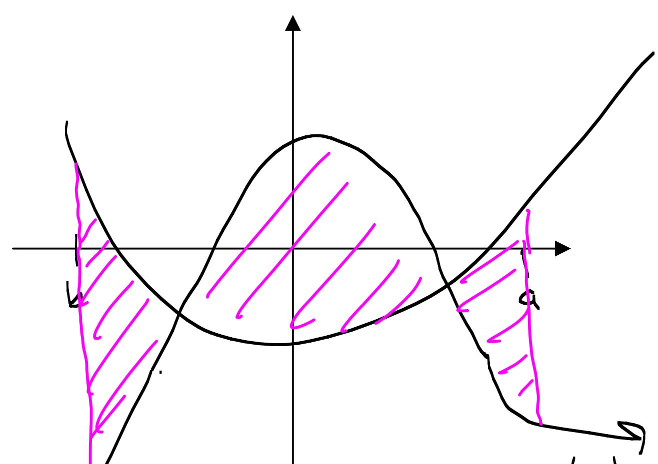
Area between curves $= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Note: If $g(x) \geq f(x)$ on $[a,b]$, then $\int_a^b [f(x) - g(x)]dx$ is negative.

Some different types of area scenarios:



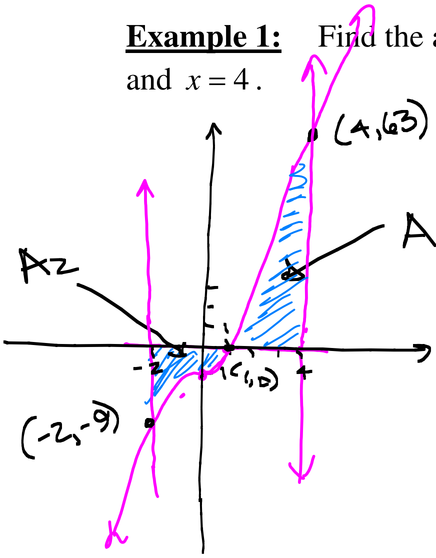
Find area enclosed by these 2 (or 3 or more) curves



Area is enclosed between 2 curves and between 2 vertical lines (endpoints)

Example 1: Find the area of the region bounded by $f(x) = x^3 - 1$ and the lines $y = 0$, $x = -2$, and $x = 4$.

x-axis



Find intersection pts:

Set $y = x^3 - 1$ and $y = 0$ equal to each other.

$$x^3 - 1 = 0 \Rightarrow x^3 = 1 \Rightarrow x = 1$$

Find where $y = x^3 - 1$ and $x = 4$ intersect:

Put in $x = 4 \Rightarrow y = (4)^3 - 1$

$y = 63$

Find where $y = x^3 - 1$ and $x = -2$ intersect:

Put in $x = -2 \Rightarrow y = (-2)^3 - 1 = -8 - 1 = -9$

$$A_1 = \int_1^4 (x^3 - 1)\,dx = \left(\frac{x^4}{4} - x\right)\Big|_1^4 = \left(\frac{4^4}{4} - 4\right) - \left(\frac{1^4}{4} - 1\right) = 64 - 4 - \frac{1}{4} + 1$$

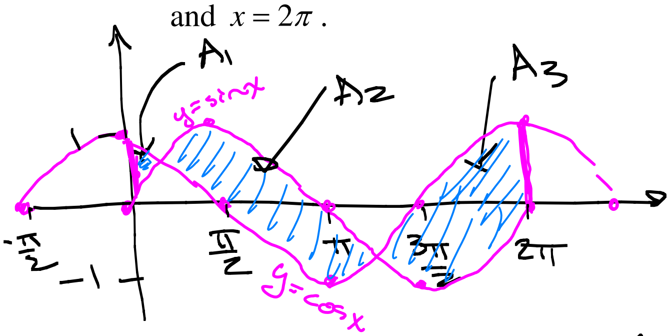
$$= 61 - \frac{1}{4} = 60.75 = \frac{243}{4}$$

$$A_2 = \int_{-2}^1 (x^3 - 1)\,dx = \left(\frac{x^4}{4} - x\right)\Big|_{-2}^1 = \left(\frac{1^4}{4} - 1\right) - \left(\frac{(-2)^4}{4} - (-2)\right) = \frac{1}{4} - 1 - 4 - 2$$

$$= \frac{1}{4} - 7 = -6.75 = -\frac{27}{4}$$

$$\text{Area} = |A_1| + |A_2| = \left|\frac{243}{4}\right| + \left|-\frac{27}{4}\right| = \frac{243}{4} + \frac{27}{4} = \frac{270}{4} = \boxed{67.50}$$

Example 2: Find the area of the region bounded by $y = \cos x$, $y = \sin x$, and the lines $x = 0$, and $x = 2\pi$.



Find intersection pts: Set y's equal:

$$\cos x = \sin x$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$A_1 = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4} = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - (\sin 0 + \cos 0)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1) = \frac{2\sqrt{2}}{2} - 1 = \sqrt{2} - 1$$

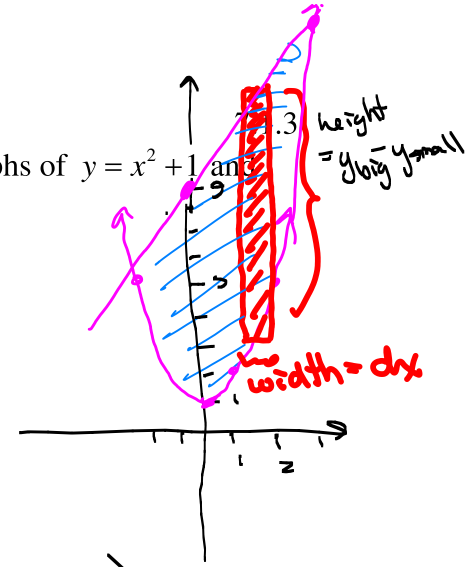
$$A_2 = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx = (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4} = -\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$

$$= -\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

$$A_3 = \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)\,dx = (\sin x + \cos x)\Big|_{\frac{5\pi}{4}}^{2\pi} = 1 + \sqrt{2}$$

$$\text{Area} = A_1 + A_2 + A_3 = \sqrt{2} - 1 + 2\sqrt{2} + 1 + \sqrt{2} = \boxed{4\sqrt{2}}$$

Example 3: Find the area of the region completely enclosed by the graphs of $y = x^2 + 1$ and $y = 2x + 9$.



Find the intersection pts: Set the y's equal:

$$x^2 + 1 = 2x + 9$$
$$x^2 - 2x - 8 = 0$$
$$(x-4)(x+2) = 0$$
$$x = 4, \; x = -2$$

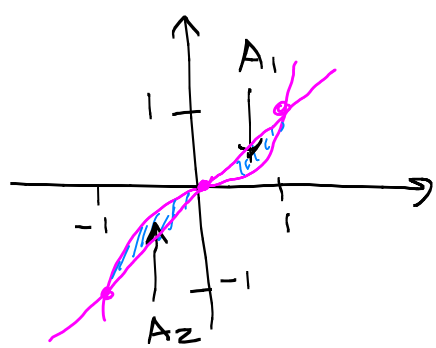
$$\text{Area} = \int_{-2}^{4} \left[(2x+9) - (x^2+1) \right] dx = \int_{-2}^{4} (-x^2 + 2x + 8)\, dx$$

$$= -\frac{x^3}{3}\Big|_{-2}^{4} + \frac{2x^2}{2}\Big|_{-2}^{4} + 8x\Big|_{-2}^{4} = \frac{-4^3}{3} - \frac{-(-2)^3}{3} + 4^2 - (-2)^2 + 8(4) - 8(-2)$$

$$= -\frac{64}{3} - \frac{8}{3} + 16 - 4 + 32 + 16 = \boxed{36}$$

Example 4: Find the area of the region completely enclosed by the graphs of $y = x^3$ and $y = x$.

$\uparrow y = 1x + 0$



Find intersection pts: Set y's equal:

$$x^3 = x$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$x(x+1)(x-1) = 0$$
$$x = 0, \; x = -1, \; x = 1$$

From symmetry, $\text{Area} = 2\int_0^1 (x - x^3)\, dx = 2\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1$

$(\text{Area} = A_1 + A_2 = 2A_1)$

$$= \left(x^2 - \frac{x^4}{2}\right)\Big|_0^1 = \left(1^2 - \frac{1^4}{2}\right) - \left(0^2 - \frac{0^4}{2}\right) = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

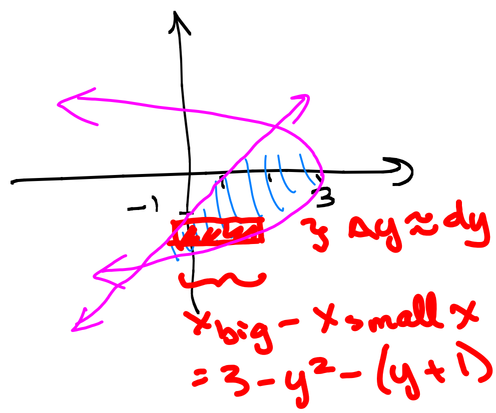
Example 5: Find the area of the region completely enclosed by the graphs of $x = y^2$ and $x = 4$.

See previous semesters' notes.

Example 6: Find the area of the region completely enclosed by the graphs of $x = 3 - y^2$ and $x = y + 1$.

$x = y + 1 \Rightarrow x - 1 = y$

$y = 1x - 1$



$$\text{Area} = \int_{-2}^{1} \left[(3 - y^2) - (y+1)\right] dy$$

$$= \int_{-2}^{1} \left[-y^2 - y + 2\right) dy$$

Find intersection pts:

Set x's equal:

$3 - y^2 = y + 1$

$0 = y^2 + y - 2$

$0 = (y+2)(y-1)$

$y = -2, \; y = 1$

$$= \frac{9}{2}$$

(see previous semesters' notes for details)