# 11.4: The Cross Product

The *cross product* (or *vector product*) of two vectors in $\mathbb{R}^3$ (3-dimensional space) yields a vector that is orthogonal to both of the vectors that produced it.

> Definition: The Cross Product
>
> Suppose that $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. The cross product of **u** and **v** is the vector
>
> $$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle.\end{aligned}$$

Note: The cross product is not defined for two-dimensional vectors.

**The determinant:**

The determinant is a concept from linear algebra. The determinant is a characteristic of square matrices, but it can help us calculate the cross product of two vectors.

The determinant of a $2 \times 2$ matrix is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ .

The determinant of a $3 \times 3$ matrix is

$$\begin{aligned}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ek - fh) - b(dk - fg) + c(dh - eg).\end{aligned}$$

**Example 1:** Find the determinant of $\begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}$.

**Example 2:** Find the determinant of $\begin{bmatrix} 2 & -7 \\ -1 & -9 \end{bmatrix}$.

**Example 3:** Find the determinant of $\begin{bmatrix} 4 & 2 & -3 \\ 7 & 5 & -8 \\ -2 & 0 & 1 \end{bmatrix}$.

**The determinant approach to calculating the cross product.**

Put the standard unit vectors **i**, **j**, and **k** in Row 1, the first vector in Row 2, and the second vector in Row 3.

The cross product of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}. \end{aligned}$$

<u>Note</u>: This is technically not a determinant, because the first row (containing **i**, **j**, and **k**) contains vectors, not scalars.

**Example 4:** Suppose $\mathbf{u} = \langle 3, 1, -2 \rangle$ and $\mathbf{v} = \langle -4, 2, 6 \rangle$. Calculate $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$. Show that the cross product is orthogonal to both of the original vectors.

**Example 5:** Suppose $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Calculate $\mathbf{u} \times \mathbf{v}$.

**Properties of the cross product:**

<u>Algebraic properties of the cross product</u>:

Let **u**, **v**, and **w** be vectors in $\mathbb{R}^3$, and let $c$ be a scalar.

1. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$
3. $c(\mathbf{u}\times\mathbf{v}) = c\mathbf{u}\times\mathbf{v} = \mathbf{u}\times c\mathbf{v}$
4. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u}$
5. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

<u>Geometric properties of the cross product</u>:

Let **u** and **v** be nonzero vectors in $\mathbb{R}^3$, and let $\theta$ be the angle between **u** and **v**. Then,

6. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** and **v**.
7. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$
8. $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if **u** and **v** are scalar multiples of each other.
9. $\|\mathbf{u}\times\mathbf{v}\|$ is the area of parallelogram having **u** and **v** as adjacent sides.

   <u>Note</u>: This means that $\frac{1}{2}\|\mathbf{u}\times\mathbf{v}\|$ is the area of a triangle having **u** and **v** as adjacent sides.

**The right-hand rule:**

The cross product follows what is known as the *right-hand rule*. This means that if you curl the fingers of your right hand from vector **u** to vector **v**, your thumb will point in the direction of $\mathbf{u}\times\mathbf{v}$.

<u>Note</u>: This means that $\mathbf{k} = \mathbf{i}\times\mathbf{j}$ .

**Example 6:** Suppose $\mathbf{u} = \langle 2, -1, 3 \rangle$ and $\mathbf{v} = \langle -4, 2, -6 \rangle$. Calculate $\mathbf{u} \times \mathbf{v}$.

**Example 7:** Find the area of the triangle with vertices $A(2,-3,4)$, $B(0,1,2)$, and $C(-1,2,0)$.

**Example 8:** Suppose the following points are the vertices of a quadrilateral. Determine whether the quadrilateral is a parallelogram. Find the area.

$A(1,1,3)$, $B(9,-1,2)$, $C(11,2,-9)$, and $D(3,4,-4)$

**The triple scalar product:**

The dot product of **u** and $\mathbf{v} \times \mathbf{w}$ is called the *triple scalar product*.

Theorem (Triple Scalar Product):

Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$. Then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is the determinant of the matrix that has **u**, **v**, and **w** as Row 1, Row 2, and Row 3, respectively.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Theorem (Volume of a Parallelepiped):

The volume of a parallelepiped with vectors **u**, **v**, and **w** as adjacent edges is

$$V = \left| \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \right|$$

**Example 9:** Find the volume of the parallelepiped having adjacent edges $\mathbf{u} = \langle 1, 3, 1 \rangle$, $\mathbf{v} = \langle 0, 6, 6 \rangle$, and $\mathbf{w} = \langle -4, 0, -4 \rangle$.

**Using the cross product to find torque:**

Suppose a force **F** is applied at the point $Q$. The torque, or moment **M** of the force **F** about a point $P$, measures the tendency of $\overrightarrow{PQ}$ to rotate counterclockwise about the point $P$.

The torque is given by

$$M = \overrightarrow{PQ} \times \mathbf{F} \ .$$

**<u>Example 10:</u>** Suppose a bolt is tightened by applying a force of 40 N to a wrench that is 0.25 m long. The force is applied at an angle of $75°$ to the axis of the wrench. Find the magnitude of the torque about the center of the bolt.