11.4: The Cross Product

The *cross product* (or *vector product*) of two vectors in $\mathbb{R}^3$ (3-dimensional space) yields a vector that is orthogonal to both of the vectors that produced it.

> Definition: The Cross Product
>
> Suppose that $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. The cross product of **u** and **v** is the vector
>
> $$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle \\ &= \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle.\end{aligned}$$

Note: The cross product is not defined for two-dimensional vectors.

The determinant:

The determinant is a concept from linear algebra. The determinant is a characteristic of square matrices, but it can help us calculate the cross product of two vectors.

The determinant of a 2×2 matrix is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

The determinant of a 3×3 matrix is

$$\begin{aligned}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ek - fh) - b(dk - fg) + c(dh - eg).\end{aligned}$$

Example 1: Find the determinant of $\begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}$.

$$\begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} = 2(9) - 5(3) = 18 - 15 = \boxed{3}$$

Example 2: Find the determinant of $\begin{bmatrix} 2 & -7 \\ -1 & -9 \end{bmatrix}$.

$$\begin{vmatrix} 2 & -7 \\ -1 & -9 \end{vmatrix} = 2(-9) - (-7)(-1) = -18 - 7 = \boxed{-25}$$

Example 3: Find the determinant of $\begin{bmatrix} 4 & 2 & -3 \\ 7 & 5 & -8 \\ -2 & 0 & 1 \end{bmatrix}$.

$$\begin{vmatrix} 4 & 2 & -3 \\ 7 & 5 & -8 \\ -2 & 0 & 1 \end{vmatrix} = 4\begin{vmatrix} 5 & -8 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 7 & -8 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 7 & 5 \\ -2 & 0 \end{vmatrix}$$

$$= 4(5-0) - 2(7-16) - 3(0-(-10))$$

$$= 20 - 2(-9) - 3(10) = 20 + 18 - 30 = \boxed{8}$$

The determinant approach to calculating the cross product.

Put the standard unit vectors **i**, **j**, and **k** in Row 1, the first vector in Row 2, and the second vector in Row 3.

The cross product of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$
$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

Note: This is technically not a determinant, because the first row (containing **i**, **j**, and **k**) contains vectors, not scalars.

Example 4: Suppose $\mathbf{u} = \langle 3, 1, -2 \rangle$ and $\mathbf{v} = \langle -4, 2, 6 \rangle$. Calculate $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$. Show that the cross product is orthogonal to both of the original vectors.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ -4 & 2 & 6 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & -2 \\ 2 & 6 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & -2 \\ -4 & 6 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 1 \\ -4 & 2 \end{vmatrix}$$

$$= 10\hat{i} - \hat{j}(18-8) + \hat{k}(6+4)$$

$$= \boxed{10\hat{i} - 10\hat{j} + 10\hat{k}}$$

Example 5: Suppose $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Calculate $\mathbf{u} \times \mathbf{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 6 & 0 \\ -2 & 1 & 1 \end{vmatrix} = \hat{i}(6-0) - \hat{j}(1-0) + \hat{k}(1+12)$$

$$= \boxed{6\hat{i} - \hat{j} + 13\hat{k}}$$

Ex 4 cont'd: Show $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

$$\vec{u} = \langle 3, 1, -2 \rangle$$

$$\vec{v} = \langle -4, 2, 6 \rangle$$

$$\vec{u} \times \vec{v} = \langle 10, -10, 10 \rangle$$

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = \langle 3, 1, -2 \rangle \cdot \langle 10, -10, 10 \rangle$$

$$= 30 - 10 - 20 = 0 \checkmark$$

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = \langle -4, 2, 6 \rangle \cdot \langle 10, -10, 10 \rangle$$

$$= -40 - 20 + 60 = 0 \checkmark$$

Note:

$$\vec{v} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 2 & 6 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \hat{i}(-4-6) - \hat{j}(8-18) + \hat{k}(-4-6)$$

$$= -10\hat{i} + 10\hat{j} - 10\hat{k} = -\vec{u} \times \vec{v}$$

Properties of the cross product:

Algebraic properties of the cross product:

Let **u**, **v**, and **w** be vectors in $\mathbb{R}^3$, and let c be a scalar.

1. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$
3. $c(\mathbf{u}\times\mathbf{v}) = c\mathbf{u}\times\mathbf{v} = \mathbf{u}\times c\mathbf{v}$
4. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \vec{0}$
5. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$
6) $\vec{u}\times\vec{u} = \vec{0}$

Geometric properties of the cross product:

Let **u** and **v** be nonzero vectors in $\mathbb{R}^3$, and let θ be the angle between **u** and **v**. Then,

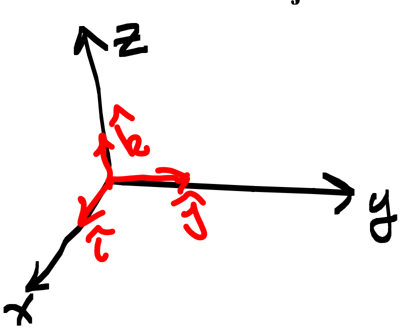
6. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** and **v**.
7. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$
8. $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if **u** and **v** are scalar multiples of each other.
9. $\|\mathbf{u}\times\mathbf{v}\|$ is the area of parallelogram having **u** and **v** as adjacent sides.

   Note: This means that $\frac{1}{2}\|\mathbf{u}\times\mathbf{v}\|$ is the area of a triangle having **u** and **v** as adjacent sides.

The right-hand rule:

The cross product follows what is known as the *right-hand rule*. This means that if you curl the fingers of your right hand from vector **u** to vector **v**, your thumb will point in the direction of $\mathbf{u}\times\mathbf{v}$.

Note: This means that $\mathbf{k} = \mathbf{i}\times\mathbf{j}$.



Example 6: Suppose $\mathbf{u} = \langle 2,-1,3 \rangle$ and $\mathbf{v} = \langle -4,2,-6 \rangle$. Calculate $\mathbf{u} \times \mathbf{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ -4 & 2 & -6 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \langle 0,0,0 \rangle = \vec{0}$$

They must be parallel!

Yes, $\vec{v} = -2\vec{u}$.

Example 7: Find the area of the triangle with vertices $A(2,-3,4)$, $B(0,1,2)$, and $C(-1,2,0)$.

$$\vec{AB} = \langle 0-2,\ 1-(-3),\ 2-4 \rangle = \langle -2, 4, -2 \rangle$$

$$\vec{AC} = \langle -1-2,\ 2-(-3),\ 0-4 \rangle = \langle -3, 5, -4 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 4 & -2 \\ -3 & 5 & -4 \end{vmatrix} = \hat{i}(-16+10) - \hat{j}(8-6) + \hat{k}(-10+12) = -6\hat{i} - 2\hat{j} + 2\hat{k}$$

Area of triangle $= \frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\sqrt{36+4+4} = \frac{1}{2}\sqrt{44}$

$$= \frac{\sqrt{44}}{2} = \frac{2\sqrt{11}}{2} = \boxed{\sqrt{11}}$$

Example 8: Suppose the following points are the vertices of a quadrilateral. Determine whether the quadrilateral is a parallelogram. Find the area.

$A(1,1,3)$, $\cancel{B(9,-1,2)}$ $B(9,-1,-2)$, $C(11,2,-9)$, and $D(3,4,-4)$

$\vec{AB} = \langle 8,-2,-5 \rangle$ ✓ $\qquad \vec{BC} = \langle 2,3,-7 \rangle$ ✓✓

$\vec{AC} = \langle 10, 1, -12 \rangle \qquad \vec{BD} = \langle -6, 5, -2 \rangle$

$\vec{AD} = \langle 2, 3, -7 \rangle$ ✓ $\qquad \vec{CD} = \langle -8, 2, 5 \rangle$ ✓

Notice: $\vec{AB} \parallel \vec{CD}$ and $\vec{BC} \parallel \vec{AD}$, so it is a parallelogram.

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & -2 & -5 \\ 2 & 3 & -7 \end{vmatrix} = \hat{i}(14+15) - \hat{j}(-56+10) + \hat{k}(24+4)$$

$$= \langle 29, +46, 28 \rangle$$

$$\|AB \times AD\| = \sqrt{29^2 + 46^2 + 28^2} = \boxed{\sqrt{3741}}$$