11.5: Lines and Planes in Space

In the plane, a slope and a point are enough to uniquely determine a line. That is not the case for lines in 3-dimensional space ($\mathbb{R}^3$).

In $\mathbb{R}^3$, we need a point and a *direction vector* **v**.

Parametric equations of a line in $\mathbb{R}^3$:

Suppose the line L passes through the point $P(x_1, y_1, z_1)$ and is parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$. Then the line can be represented by the following parametric equations:

$$x = x_1 + at$$
$$y = y_1 + bt$$
$$z = z_1 + ct$$

The scalars *a*, *b*, and *c* are sometimes called the *direction numbers* for the line.

If *a*, *b*, and *c* are all nonzero, the line can also be represented by the symmetric equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Example 1: Find the parametric and symmetric equations for the line that passes through the point $P(4, 1, -3)$ and is parallel to $\mathbf{v} = -2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$. Find two additional points on the line.

Example 2: Find parametric and symmetric equations for the line that passes through the points $P(-3,0,2)$ and is parallel to $\mathbf{v}=\langle 0,6,3\rangle$. Find two additional points on the line.

Example 3: Find parametric and symmetric equations for the line that passes through the points $P(2,0,2)$ and $Q(1,4,-3)$.

Example 4: Find parametric and symmetric equations for the line that passes through the points $P(-3,5,4)$ and is parallel to the line with symmetric equations $\dfrac{x-1}{3}=\dfrac{y+1}{-2}=z-3$.

Equations of planes in $\mathbb{R}^3$:

To determine the equation of a line in $\mathbb{R}^2$, we need a point on the line and a slope.
To determine the equation of a line in $\mathbb{R}^3$, we need a point on the line and a direction vector.
To determine the equation of a plane in $\mathbb{R}^3$, we need a point on the plane and a vector that is *normal* to the plane (it forms an angle of $90°$ with any vector in the plane).

<u>Standard Form for the Equation of a Plane</u>:

Suppose a plane contains the point $P(x_1, y_1, z_1)$ and has normal vector $\mathbf{n} = \langle a, b, c \rangle$. Then the *standard form* for the equation of the plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 .$$

This equation can be rearranged into the form $ax + by + cz + d = 0$, which is called the *general form* for the equation of the plane.

<u>Example 5:</u> Find an equation of the plane that includes the points $(3, -1, 2)$, $(2, 1, 5)$, and $(1, -2, -2)$.

Example 6: Suppose a line passes through the point $(-4,5,2)$ and is normal to the plane with equation $-x+2y+z=5$. Find a set of parametric equations for the line. Also, find the point where the line intersects the plane.

Example 7: Find an equation for the plane that passes through the point $(3,2,2)$ and is perpendicular to the line given by $\frac{x-1}{4}=y+2=\frac{z+3}{-3}$. Write the equation of the plane in general form.

Example 8: Find an equation for the plane that passes through the point $(1,2,3)$ and is parallel to the *yz*-plane.

Example 9: Find the point of intersection of the plane with equation $2x+3y=-5$, and the line given by $\frac{x-1}{4}=\frac{y}{2}=\frac{z-3}{6}$.

Example 10: Find the point of intersection of the plane with equation $-27x-19y+7z=2$, and the line given by equations $x=1+2t$, $y=4-t$, and $z=3+5t$.

Angle between two planes:

The angle between two planes is the same as the angle between their normal vectors. Therefore, if vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to two intersecting planes, the angle between the two planes is described by this equation:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$$

Thus, the planes are

- orthogonal (perpendicular) when $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$
- parallel when $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$.

<u>Example 11:</u> Two planes have equations $3x + y - 4z = 3$ and $-9x - 3y + 12z = 4$. Determine if the planes are parallel, orthogonal, or neither. If neither, find the angle between them.

<u>Example 12:</u> Two planes have equations $x + y + z = 1$ and $x - 2y + 3z = 1$. Determine if the planes are parallel, orthogonal, or neither. If neither, find the angle between them. Find the equation of the line of intersection.

Distance between a point and a plane:

Theorem: Suppose P is a point in the plane, $\mathbf{n}$ is normal to the plane, and the point Q is not in the plane. Then the distance between a plane and the point Q is

$$D = \left\| \operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} \right\| = \frac{\left| \overrightarrow{PQ} \cdot \mathbf{n} \right|}{\|\mathbf{n}\|}$$

Example 13: Find the distance between the point $(1,3,-1)$ and the plane with equation $3x - 4y + 5z = 6$.

Example 14: Find the distance between the point (x_0, y_0, z_0) and the plane with equation $ax + by + cz + d = 0$.

<u>Example 15:</u> Verify that the planes given by $2x+3y-4z=2$ and $4x+6y-8z=27$ are parallel. Then find the distance between them.

Distance between a point and a line in $\mathbb{R}^3$:

<u>Theorem</u>: The distance between a point Q and a line in $\mathbb{R}^3$ is

$$D=\frac{\left\|\overrightarrow{PQ}\times\mathbf{u}\right\|}{\|\mathbf{u}\|}$$

where $\mathbf{u}$ is a direction vector for the line and P is a point on the line.

<u>Example 16:</u> Find the distance between the point $Q(1,-2,4)$ and the line given by $x=2t$, $y=t-3$, $z=2t+2$.