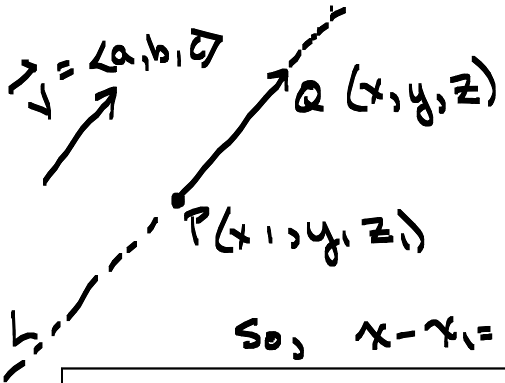
11.5: Lines and Planes in Space

In the plane, a slope and a point are enough to uniquely determine a line. That is not the case for lines in 3-dimensional space ($\mathbb{R}^3$).

In $\mathbb{R}^3$, we need a point and a *direction vector* **v**.



Note: $\overrightarrow{PQ} = t\vec{v}$, where t is a scalar

$\langle x-x_1, y-y_1, z-z_1 \rangle = t\langle a,b,c \rangle$
$= \langle ta, tb, tc \rangle$

So, $x - x_1 = ta$, $y - y_1 = tb$, $z - z_1 = tc$

Parametric equations of a line in $\mathbb{R}^3$:

Suppose the line L passes through the point $P(x_1, y_1, z_1)$ and is parallel to the vector $\mathbf{v} = \langle a,b,c \rangle$. Then the line can be represented by the following parametric equations:

$$\left.\begin{aligned} x &= x_1 + at \\ y &= y_1 + bt \\ z &= z_1 + ct \end{aligned}\right\}$$

solve these for t:

$$\left.\begin{aligned} t &= \frac{x-x_1}{a} \\ t &= \frac{y-y_1}{b} \\ t &= \frac{z-z_1}{c} \end{aligned}\right\}$$

The scalars *a*, *b*, and *c* are sometimes called the *direction numbers* for the line.

If *a*, *b*, and *c* are all nonzero, the line can also be represented by the symmetric equations

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}.$$

Example 1: Find the parametric and symmetric equations for the line that passes through the point $P(4,1,-3)$ and is parallel to $\mathbf{v} = -2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$. Find two additional points on the line.

$(x_1, y_1, z_1) = (4, 1, -3)$, $\vec{v} = \langle a,b,c \rangle = \langle -2, -1, 7 \rangle$

$$\boxed{\begin{aligned} x &= 4 - 2t \\ y &= 1 - t \\ z &= -3 + 7t \end{aligned}}$$

To find the symmetric eqns, solve for t:

$x = 4 - 2t \Rightarrow \frac{x-4}{-2} = t$

$y = 1 - t \Rightarrow \frac{y-1}{-1} = t$

$z = -3 + 7t \Rightarrow \frac{z+3}{7} = t$

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Symmetric eqns: $\boxed{\dfrac{x-4}{-2} = \dfrac{y-1}{-1} = \dfrac{z+3}{7}}$

To find points on the line, choose values for t.

Example 2: Find parametric and symmetric equations for the line that passes through the points $P(-3,0,2)$ ~~and is parallel to $v = \langle 0,6,3 \rangle$. Find two additional points on the line.~~

and the point $Q(1,2,3)$

Use $\overrightarrow{PQ}$ as the direction vector:

$$\overrightarrow{PQ} = \langle 4, 2, 1 \rangle$$

Write parametric eqns starting with either known point:

$$\boxed{\begin{aligned} x &= -3 + 4t \\ y &= 0 + 2t \\ z &= 2 + 1t \end{aligned}}$$

~~**Example 3:** Find parametric and symmetric equations for the line that passes through the points $P(2,0,2)$ and $Q(1,4,-3)$.~~

Ex: Find parametric eqns and symmetric eqns for the line containing $P(-3,0,2)$ and parallel to $\vec{v} = \langle 0,6,3 \rangle$

$$\left.\begin{aligned} x &= -3 + 0t \\ y &= 0 + 6t \\ z &= 2 + 3t \end{aligned}\right\} \quad \boxed{\begin{aligned} x &= -3 \\ y &= 6t \\ z &= 2 + 3t \end{aligned}}$$

Solve y and z for t:

$t = \dfrac{y}{6}$

$t = \dfrac{z-2}{3}$

Symmetric eqns: $\boxed{\dfrac{y}{6} = \dfrac{z-2}{3}, \quad x = -3}$

Example 4: Find parametric and symmetric equations for the line that passes through the points $P(-3,5,4)$ and is parallel to the line with symmetric equations $\dfrac{x-1}{3} = \dfrac{y+1}{-2} = z - 3$.

Use the symmetric eqns to get the direction vector.

set $t = \dfrac{x-1}{3} = \dfrac{y+1}{-2} = \dfrac{z-3}{1}$

$3t + 1 = x, \quad -2t - 1 = y, \quad t + 3 = z$

So direction vector is $\vec{v} = \langle 3, -2, 1 \rangle$

So, for the desired line,

$x = -3 + 3t$

$y = 5 - 2t, \quad z = 4 + 1t.$

Equations of planes in $\mathbb{R}^3$:

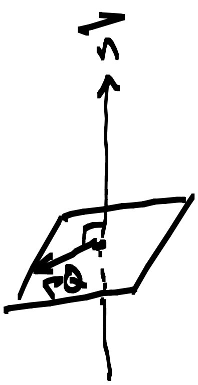
To determine the equation of a line in $\mathbb{R}^2$, we need a point on the line and a slope.
To determine the equation of a line in $\mathbb{R}^3$, we need a point on the line and a direction vector.
To determine the equation of a plane in $\mathbb{R}^3$, we need a point on the plane and a vector that is *normal* to the plane (it forms an angle of $90°$ with any vector in the plane).

<u>Standard Form for the Equation of a Plane</u>:

Suppose a plane contains the point $P(x_1, y_1, z_1)$ and has normal vector $\mathbf{n} = \langle a, b, c \rangle$. Then the *standard form* for the equation of the plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This equation can be rearranged into the form $ax + by + cz + d = 0$, which is called the *general form* for the equation of the plane.



Suppose $P = (x_1, y_1, z_1)$ and $Q = (x, y, z)$ are in the plane

$\vec{n} = \langle a, b, c \rangle$

$\vec{PQ} = \langle x - x_1, y - y_1, z - z_1 \rangle$

$\vec{n} \perp \vec{PQ} \Rightarrow \vec{n} \cdot \vec{PQ} = 0 \Rightarrow \langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

<u>Example 5:</u> Find an equation of the plane that includes the points $(3, -1, 2)$, $(2, 1, 5)$, and $(1, -2, -2)$.

$P(3, -1, 2)$

$Q(2, 1, 5)$

$R(1, -2, -2)$

$\vec{PQ} = \langle -1, 2, 3 \rangle$

$\vec{PR} = \langle -2, -1, -4 \rangle$

Cross them to get a normal vector.

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 3 \\ -2 & -1 & -4 \end{vmatrix} = \hat{i}(-5) - \hat{j}(10) + \hat{k}(5) = \langle -5, -10, 5 \rangle$$

Using $P = (3, -1, 2)$ as starting point,

$-5(x - 3) - 10(y + 1) + 5(z - 2) = 0$

Write in general form: $-5x + 15 - 10y - 10 + 5z - 10 = 0$

$-5x - 10y + 5z - 5 = 0$

Divide by -5: $\boxed{x + 2y - z + 1 = 0}$

Example 6: Suppose a line passes through the point $(-4, 5, 2)$ and is normal to the plane with equation $-x+2y+z=5$. Find a set of parametric equations for the line. Also, find the point where the line intersects the plane.

direction vector for line = normal vector for plane $= \langle -1, 2, 1 \rangle$

Parametric eqns for line:

$$\boxed{\begin{aligned} x &= -4-t \\ y &= 5+2t \\ z &= 2+t \end{aligned}}$$

$-x+2y+z=5$

$-(-4-t)+2(5+2t)+(2+t)=5$

$4+t+10+4t+2+t=5$

$6t+16=5$

$6t=-11 \Rightarrow t=-\frac{11}{6}$

To find pt. where line intersects plane, substitute these parametric representations of x, y, z into eqn of plane.

From above calculation, $t=-\frac{11}{6}$, then $x=-4-(-\frac{11}{6})=-13/6$

$y=5+2(-\frac{11}{6})=\frac{8}{6}$, $z=2-\frac{11}{6}=\frac{1}{6}$

$\boxed{\text{Point is } (-\frac{13}{6}, \frac{4}{3}, \frac{1}{6})}$

Example 7: Find an equation for the plane that passes through the point $(3, 2, 2)$ and is perpendicular to the line given by $\underbrace{\frac{x-1}{4}=y+2=\frac{z+3}{-3}}$. Write the equation of the plane in general form.

$\vec{n}=\langle 4, 1, -3 \rangle$

$4(x-3)+1(y-2)-3(z-2)=0$

Works out to $4x+y-3z-8=0$

Example 8: Find an equation for the plane that passes through the point $(1, 2, 3)$ and is parallel to the *yz*-plane.

We can use $\vec{i}=\langle 1, 0, 0 \rangle$ as a normal vector.

$1(x-1)+0(y-2)+0(z-3)=0$

$x-1=0$

$\boxed{x=1}$

Example 9: Find the point of intersection of the plane with equation $2x+3y=-5$, and the line given by $\frac{x-1}{4}=\frac{y}{2}=\frac{z-3}{6}$.

$$t = \frac{x-1}{4} = \frac{y}{2} = \frac{z-3}{6}$$

Note:
$\vec{n}\cdot\vec{v} = \langle 2,3,0\rangle \cdot \langle 4,2,6\rangle$
$= 8+6 = 14 \neq 0$
(not orthogonal). Line must intersect plane at 1 point

$x = 4t+1$
$y = 2t$
$z = 6t+3$
Parametric eqns for line.

Substitute these into the eqn of the given plane:

$2x+3y=-5$
$2(4t+1)+3(2t)=-5$
$8t+2+6t=-5$
$14t=-7$
$t=-\frac{1}{2}$

Put $t=-\frac{1}{2}$ into parametric eqns:
$x = 4(-\frac{1}{2})+1 = -2+1 = -1$
$y = 2(-\frac{1}{2}) = -1$
$z = 6(-\frac{1}{2})+3 = -3+3 = 0$

$\boxed{\text{Intersection point: } (-1,-1,0)}$

Example 10: Find the point of intersection of the plane with equation $-27x-19y+7z=2$, and the line given by equations $x=1+2t$, $y=4-t$, and $z=3+5t$.

$-27(1+2t)-19(4-t)+7(3+5t)=2$
$-27-54t-76+19t+21+35t=2$
$-82+0t=2$
$-82=2$ False! Whoa, what's wrong?

Direction vector for line: $\vec{v} = \langle 2,-1,5\rangle$
Normal vector for plane: $\vec{n} = \langle -27,-19,7\rangle$
$\vec{n}\cdot\vec{v} = -54+19+35 = 0$. So $\vec{n}\perp\vec{v}$ ($\vec{n}$ and $\vec{v}$ are orthogonal).
So $\vec{v}$ lives in the plane (or could be placed in the plane)
thus the line either lies entirely in the plane, or the line never intersects the plane.

Point on line ($t=0$): $(1,4,3)$. Is it in plane?
$-27(1)-19(4)+7(3)=2$
$-82=2$ False. Pt $(1,4,3)$ is not in the plane.

$\boxed{\text{This line never intersects this plane.}}$

Angle between two planes:

The angle between two planes is the same as the angle between their normal vectors. Therefore, if vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to two intersecting planes, the angle between the two planes is described by this equation:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} \qquad \theta = \cos^{-1}\left(\frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\|\vec{n}_2\|}\right)$$

Thus, the planes are

- orthogonal (perpendicular) when $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$
- parallel when $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$.

Example 11: Two planes have equations $3x + y - 4z = 3$ and $-9x - 3y + 12z = 4$. Determine if the planes are parallel, orthogonal, or neither. If neither, find the angle between them.

$\vec{n}_1 = \langle 3, 1, -4 \rangle$

$\vec{n}_2 = \langle -9, -3, 12 \rangle$

$\vec{n}_2 = -3\vec{n}_1$, so the $\boxed{\text{planes are parallel.}}$

Example 12: Two planes have equations $x + y + z = 1$ and $x - 2y + 3z = 1$. Determine if the planes are parallel, orthogonal, or neither. If neither, find the angle between them. Find the equation of the line of intersection.

$\vec{n}_1 = \langle 1, 1, 1 \rangle$, $\vec{n}_2 \langle 1, -2, 3 \rangle$

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\|\vec{n}_2\|} = \frac{|1-2+3|}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}}$$

$$\boxed{\theta \approx 72.02^\circ}$$

To find the line of intersection, we need a direction and a point:

Direction vector for line: $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$

$= \hat{i}(5) - \hat{j}(2) + \hat{k}(-3) = \langle 5, -2, -3 \rangle$

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Find a point that is on both planes:

$$\left.\begin{aligned} x+y+z&=1 \\ x-2y+3z&=1 \end{aligned}\right\}$$

set $z=0$: $x+y=1$

$x-2y=1$

subtract eqns: $3y=0$

$y=0$

$z=0,\ y=0 \Rightarrow x+y+z=1$ becomes $x=1$

Point: $(1,0,0)$

Direction vector for line: $\langle 5,-2,-3\rangle$

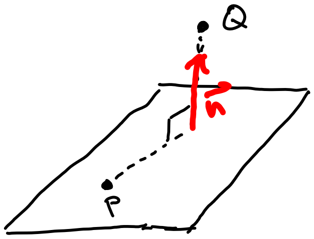
Parametric eqns for line: $\left.\begin{aligned} x&=1+5t \\ y&=0-2t \\ z&=0-3t \end{aligned}\right\}$ $\boxed{\begin{aligned} x&=1+5t \\ y&=-2t \\ z&=-3t \end{aligned}}$

Symmetric eqns: solve for t:

$$\boxed{\frac{x-1}{5}=\frac{y}{-2}=\frac{z}{-3}}$$

Distance between a point and a plane:

> Theorem: Suppose P is a point in the plane, $\mathbf{n}$ is normal to the plane, and the point Q is not in the plane. Then the distance between a plane and the point Q is
>
> $$D = \left\|\text{proj}_{\mathbf{n}} \overrightarrow{PQ}\right\| = \frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}$$



Example 13: Find the distance between the point $\overset{Q}{(1,3,-1)}$ and the plane with equation $3x-4y+5z=6$.

$\vec{n} = \langle 3, -4, 5\rangle$

Find any point in the plane:

Let $y=0,\ z=0$: $\quad 3x=6$

$x=2$

Use point $P(2,0,0)$ on plane

$$D = \frac{|\overrightarrow{PQ}\cdot\vec{n}|}{\|\vec{n}\|} = \frac{|\langle -1, 3, -1\rangle\cdot\langle 3,-4,5\rangle|}{\sqrt{9+16+25}} = \frac{|-20|}{\sqrt{50}} = \frac{20}{\sqrt{50}} = \frac{20}{5\sqrt{2}} = \boxed{\frac{4}{\sqrt{2}}}$$

Example 14: Find the distance between the point $\overset{Q}{(x_0, y_0, z_0)}$ and the plane with equation $ax+by+cz+d=0$.

$\vec{n} = \langle a, b, c\rangle$ Suppose $P(x,y,z)$ is a point in the plane.

Then $\overrightarrow{PQ} = \langle x_0-x, y_0-y, z_0-z\rangle$

$$D = \frac{|\overrightarrow{PQ}\cdot\vec{n}|}{\|\vec{n}\|} = \frac{|\langle x_0-x, y_0-y, z_0-z\rangle\cdot\langle a,b,c\rangle|}{\sqrt{a^2+b^2+c^2}} = \frac{|a(x_0-x)+b(y_0-y)+c(z_0-z)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_0+by_0+cz_0-ax-by-cz|}{\sqrt{a^2+b^2+z^2}} = \frac{|ax_0+by_0+cz_0-\overbrace{(ax+by+cz)}^{-d}|}{\sqrt{a^2+b^2+c^2}}$$

$\therefore$ The distance between point $Q(x_0, y_0, z_0)$ and plane $ax+by+cz+d=0$ is $\boxed{\dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}}$

$4x+6y-8z-27=0$

Example 15: Verify that the planes given by $2x+3y-4z=2$ and $4x+6y-8z=27$ are parallel. Then find the distance between them.

Are the normal vectors scalar multiples of each other?

$\vec{n}_1 = \langle 2,3,-4\rangle$, $\vec{n}_2 = \langle 4,6,-8\rangle$

$\vec{n}_2 = 2\vec{n}_1$, so the planes are parallel.

Find a point in one of the planes: $(1,0,0)$ is on 1st plane.

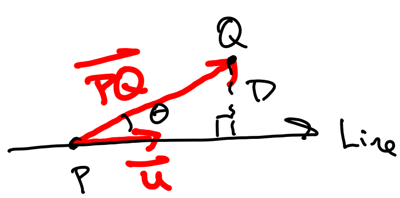
Use formula from previous example:

$$D = \frac{|ax_0+by_0+cz_0+D|}{\|\sqrt{a^2+b^2+c^2}\|} = \frac{|4(1)+6(0)-8(0)-27|}{\sqrt{4^2+6^2+8^2}} = \frac{|-23|}{\sqrt{116}} = \boxed{\frac{23}{\sqrt{116}}}$$

$$\approx \boxed{2.13550}$$

Distance between a point and a line in $\mathbb{R}^3$:

> Theorem: The distance between a point Q and a line in $\mathbb{R}^3$ is
>
> $$D = \frac{\|\overrightarrow{PQ}\times\mathbf{u}\|}{\|\mathbf{u}\|}$$
>
> where $\mathbf{u}$ is a direction vector for the line and P is a point on the line.



$D = \|\overrightarrow{PQ}\|\sin\theta$

Also, $\|\vec{u}\|\,\|\overrightarrow{PQ}\|\sin\theta = \|\vec{u}\times\overrightarrow{PQ}\|$

$$\sin\theta = \frac{\|\vec{u}\times\overrightarrow{PQ}\|}{\|\vec{u}\|\,\|\overrightarrow{PQ}\|}$$

$$D = \|\overrightarrow{PQ}\|\sin\theta = \frac{\|\vec{u}\times\overrightarrow{PQ}\|}{\|\vec{u}\|\,\|\overrightarrow{PQ}\|}\|\overrightarrow{PQ}\| = \frac{\|\vec{u}\times\overrightarrow{PQ}\|}{\|\vec{u}\|}$$

Example 16: Find the distance between the point $Q(1,-2,4)$ and the line given by $x=2t$, $y=t-3$, $z=2t+2$.

$\vec{u} = \langle 2,1,2\rangle$ direction vector for line

To find a P on line, let $t=0$: $P = (2(0), 0-3, 2(0)+2) = (0,-3,2)$

$\overrightarrow{PQ} = \langle 1,1,2\rangle$

$$\vec{u}\times\overrightarrow{PQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix} = \hat{i}(0) - \hat{j}(2) + \hat{k}(1) = \langle 0,-2,1\rangle$$

$$D = \frac{\|\vec{u}\times\overrightarrow{PQ}\|}{\|\vec{u}\|} = \frac{\sqrt{0^2+2^2+1^2}}{\sqrt{2^2+1^2+2^2}} = \frac{\sqrt{5}}{\sqrt{9}} = \boxed{\frac{\sqrt{5}}{3}}$$