11.6: Surfaces in Space

You should be familiar with 3 types of surfaces in $\mathbb{R}^3$:

1. Cylindrical surfaces
2. Quadric surfaces
3. Surfaces of revolution

Cylindrical surfaces (cylinders):

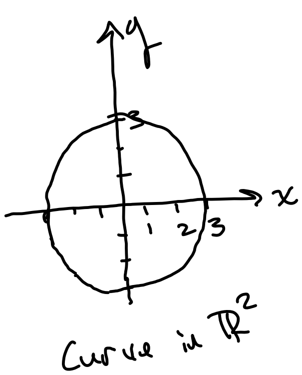
Definition:

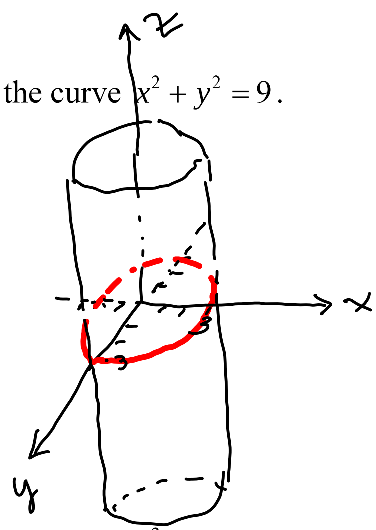
Suppose C is a curve in a plane, and L is a line not in a parallel plane. The set of all lines parallel to L and intersecting C is a *cylindrical surface*, or *cylinder*.

The curve C is the *generating curve* (or the *directrix*) of the cylinder, and the parallel lines are the *rulings*.

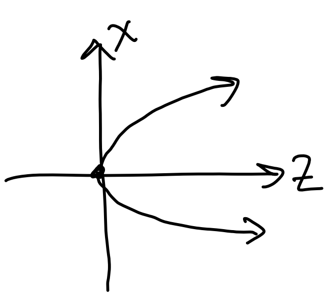
In a *right cylinder*, the rulings are perpendicular to the plane containing C. In this class, we'll stick to right cylinders in which C is contained in one of the coordinate planes. (Thus, the rulings are parallel to one of the coordinate axes.)

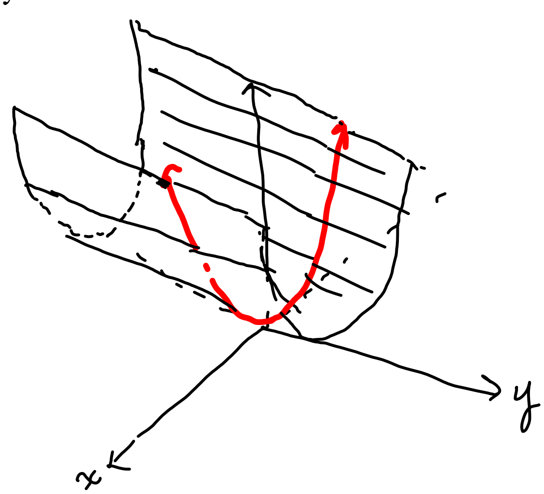
Example 1: Sketch the cylinder generated by the curve $x^2 + y^2 = 9$.



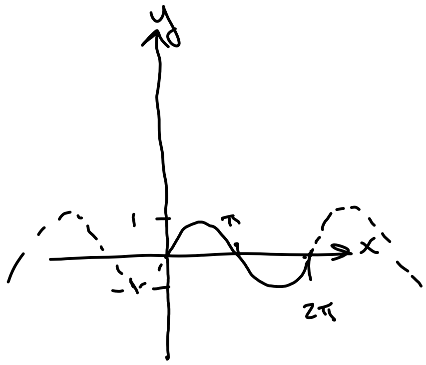


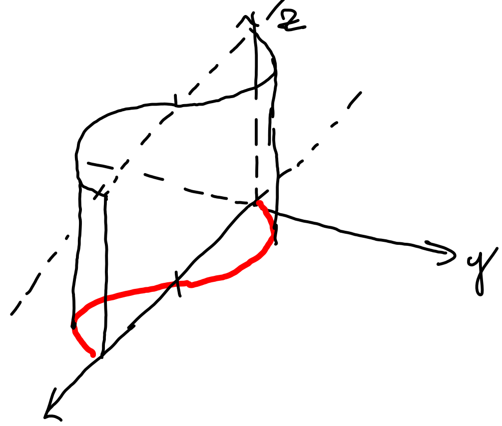
Example 2: Sketch the cylinder generated by the curve $x^2 - z = 0$.





Example 3: Sketch the cylinder generated by the curve $y = \sin x$.





Example 4: Sketch the cylinder generated by the curve $x^2 + y^2 - 2x = 0$.

$x^2 - 2x + y^2 = 0$

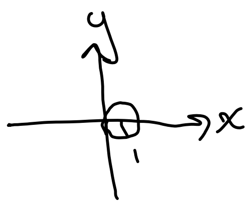
Complete the square:

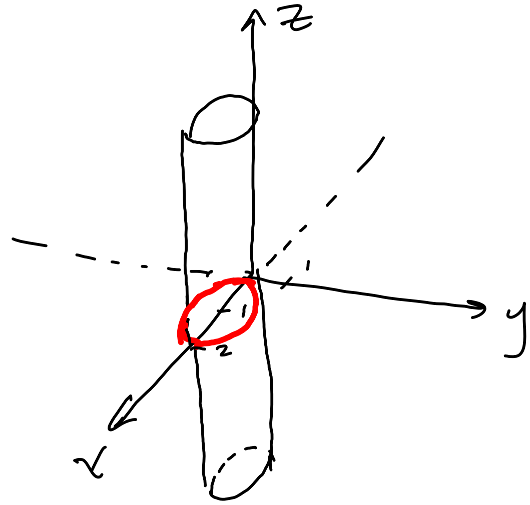
$(x^2 - 2x + 1) + y^2 = 1$

$\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$

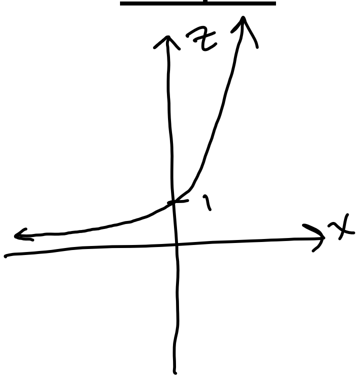
$(x-1)^2 + y^2 = 1$

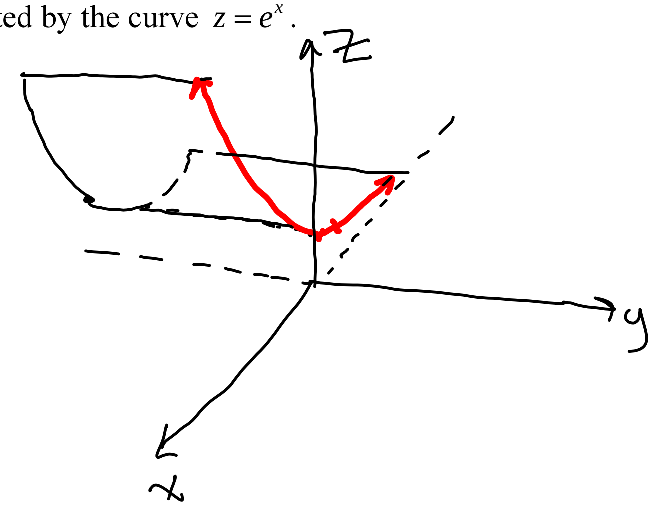
Circle, radius 1, center $(1, 0)$





Example 5: Sketch the cylinder generated by the curve $z = e^x$.





Quadric surfaces:

Quadric surfaces in $\mathbb{R}^3$ are the three-dimensional analogs of conic sections in $\mathbb{R}^2$.

The *trace* of a surface in a plane is the intersection of the surface with the plane. The traces of the quadric surfaces are conics (ellipses, hyperbolas, and parabolas).

A quadric surface in $\mathbb{R}^3$ is described by a second-degree equation in three variables. The general form of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

In this class, we'll stick to surfaces that are oriented in the same direction as one of the coordinate axes (no rotated conics!).

6 basic types of quadric surfaces:

1. Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

2. Hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

3. Hyperboloid of two sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

4. Elliptic cone:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0 \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

5. Elliptic paraboloid:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad y = \frac{x^2}{a^2} + \frac{z^2}{c^2} \qquad x = \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

6. Hyperbolic paraboloid:

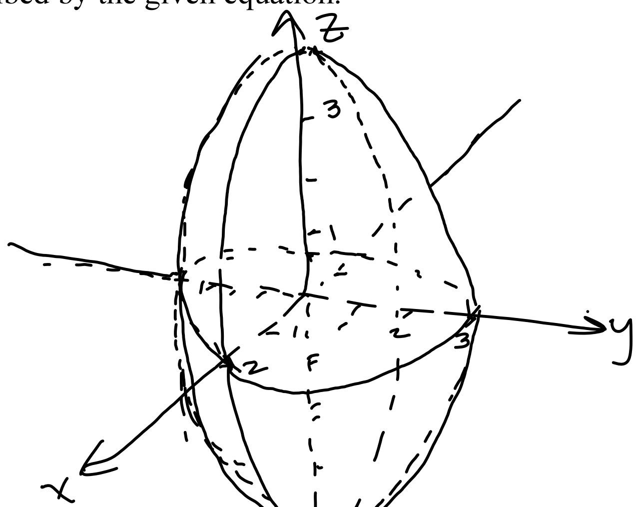
$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \qquad y = \frac{x^2}{a^2} - \frac{z^2}{c^2} \qquad x = \frac{y^2}{b^2} - \frac{z^2}{c^2}$$

$$z = -\frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad y = -\frac{x^2}{a^2} + \frac{z^2}{c^2} \qquad x = -\frac{y^2}{b^2} + \frac{z^2}{c^2}$$

See Pages 796–797 for accurate pictures of quadric surfaces. To practice sketching them by hand, see the worksheet handout. Trace it if you need to. Trust me, it helps! (Sincere appreciation to Mr. Bob Collings for the good hand drawings.)

Example 6: Sketch the surface described by the given equation.

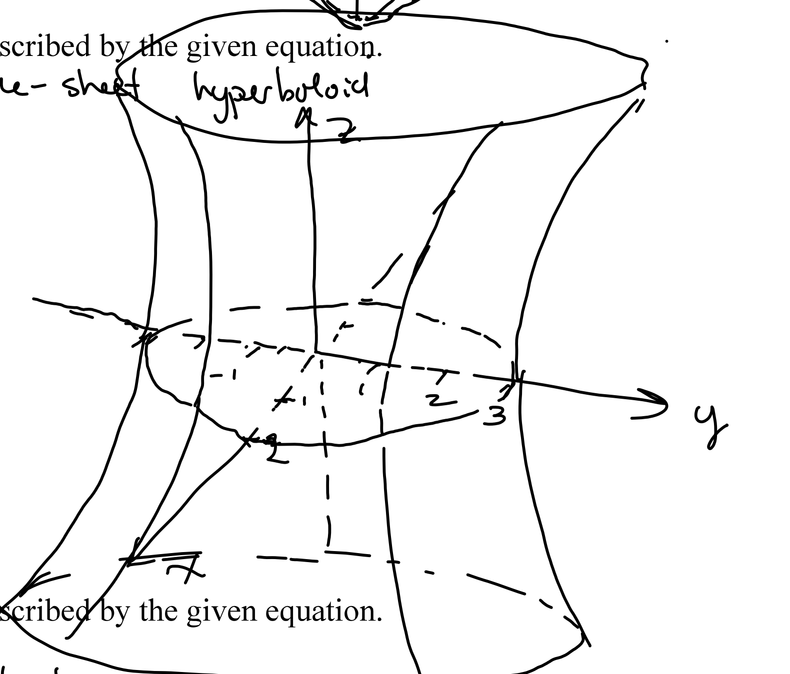
$$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1$$



Example 7: Sketch the surface described by the given equation.

one-sheet hyperboloid

$$\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$$



Traces:
(with coordinate planes)

xy: (z=0) $\frac{x^2}{4}+\frac{y^2}{9}=1$ ellipse

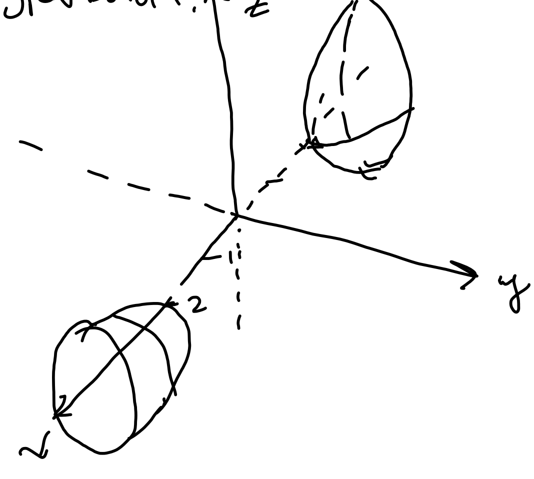
yz: (x=0) $\frac{y^2}{9}-\frac{z^2}{16}=1$ hyperbola

xz: (y=0) $\frac{x^2}{9}-\frac{z^2}{16}=1$ hyperbola

Example 8: Sketch the surface described by the given equation.

$$\frac{x^2}{4}-\frac{y^2}{9}-\frac{z^2}{16}=1$$

2-sheet hyperboloid



Traces:

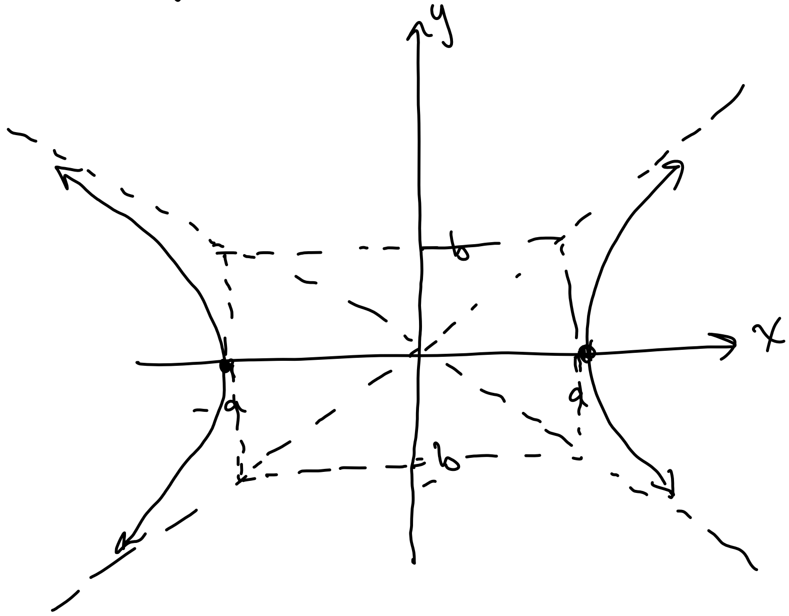
xy plane: (z=0) $\frac{x^2}{4}-\frac{y^2}{9}=1$ hyperbola

yz plane: (x=0) $-\frac{y^2}{9}-\frac{z^2}{16}=1$ empty set impossible

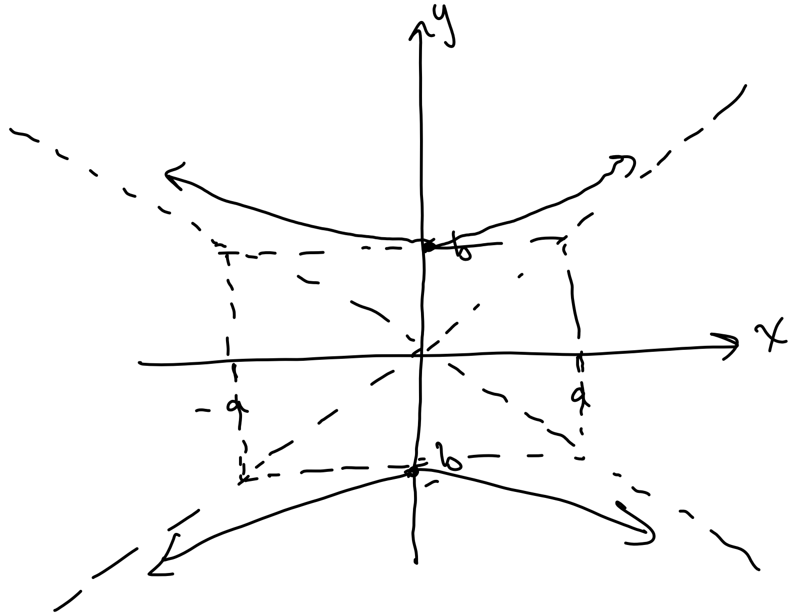
xz plane (y=0) $\frac{x^2}{4}-\frac{z^2}{16}=1$ hyperbola

Review: Graphing hyperbolas in $\mathbb{R}^2$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

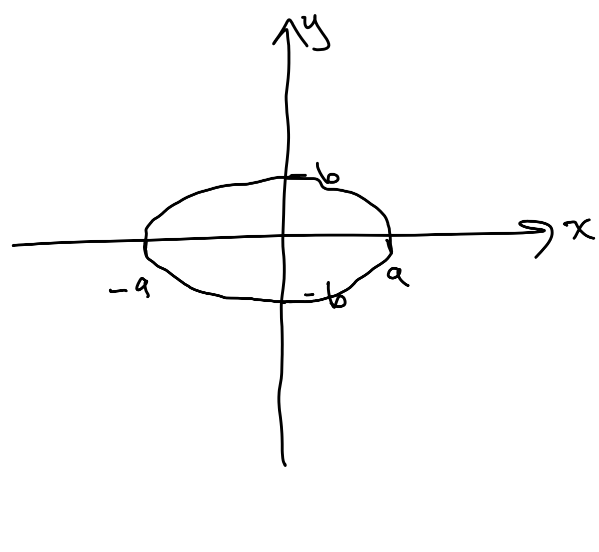


$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

ellipse



Example 9: Sketch the surface described by the given equation.

$$z^2 - x^2 - y^2 = 1$$

2-sheet hyperboloid

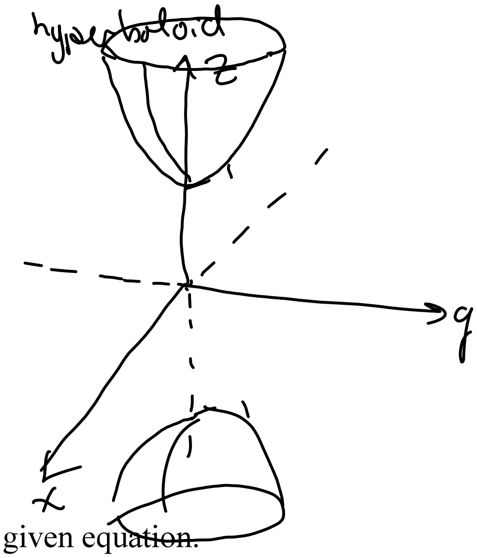
$$\frac{z^2}{1} - \frac{x^2}{1} - \frac{y^2}{1} = 1$$

Traces:

xy: ($z=0$) $-x^2 - y^2 = 1$ empty set

yz: ($x=0$) $z^2 - y^2 = 1$ hyp

xz: ($y=0$) $z^2 - x^2 = 1$ hyp



Example 10: Sketch the surface described by the given equation.

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 0$$

elliptic cone around z-axis

Traces:

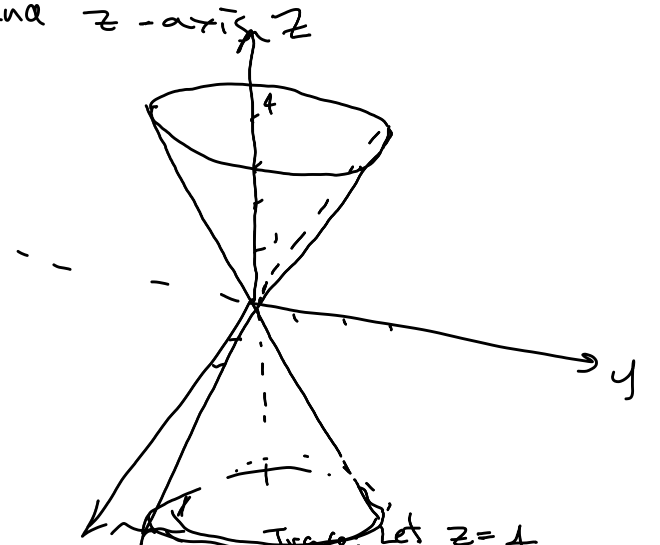
xy: ($z=0$) $\frac{x^2}{4} + \frac{y^2}{9} = 0$ pt. $(0,0,0)$

yz: ($x=0$) $\frac{y^2}{9} - \frac{z^2}{16} = 0$

$y^2 = \frac{9z^2}{16}$

$y = \pm \frac{3}{4} z$ lines

xz: ($y=0$) $\frac{x^2}{4} - \frac{z^2}{16} = 0 \Rightarrow x = \pm \frac{1}{2} z$ lines



Trace: Let $z = 4$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

ellipse

Example 11: Sketch the surface described by the given equation.

$$z = x^2 + y^2$$

elliptic paraboloid

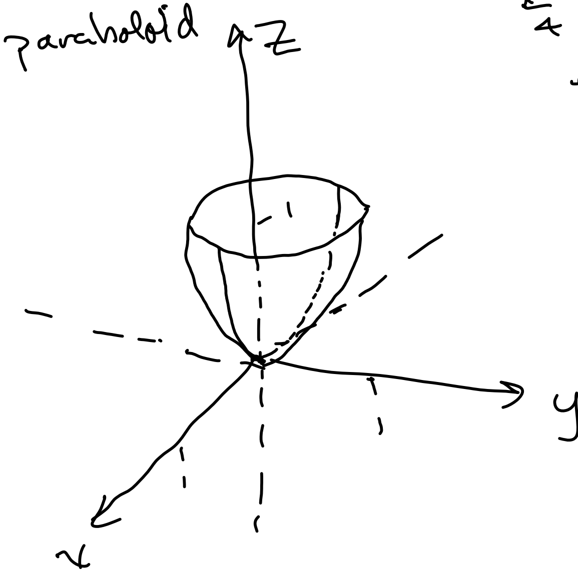
Traces:

xy: ($z=0$) $0 = x^2 + y^2 \Rightarrow$ pt $(0,0,0)$

yz: ($x=0$) $z = y^2$ parabola

xz: ($y=0$) $z = x^2$ parabola

$z=1$: $1 = x^2 + y^2$ circle



Example 12: Sketch the surface described by the given equation.

$$z = x^2 - y^2$$

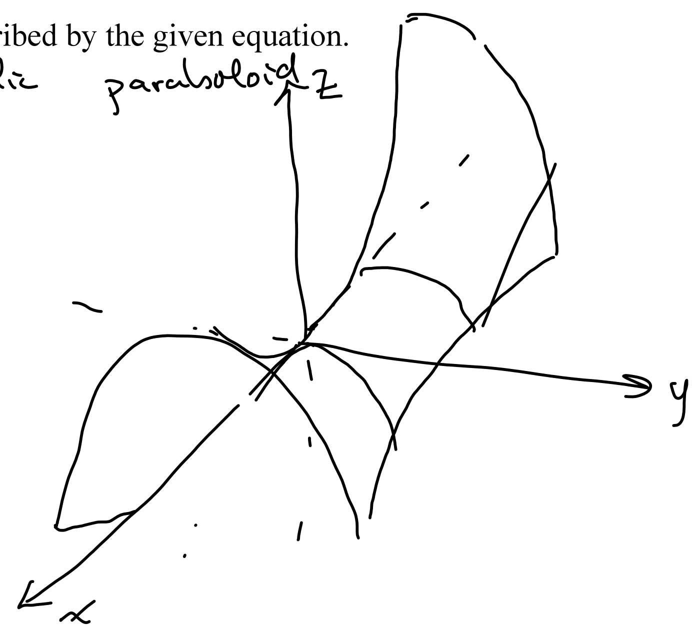
hyperbolic paraboloid

Traces:

xy: $(z=0)$ $\quad 0 = x^2 - y^2$ lines $\quad y = \pm x$

xz $(y=0)$ $\quad z = x^2$ parabola

yz $(x=0)$ $\quad z = -y^2$ parabola



Example 13: Sketch the surface described by the given equation.

$$z = y^2 - x^2$$

Surfaces of revolution:

You found volumes of solids of revolution in Calculus 1 or Calculus 2; you found surface area of a surface of revolution. But what is the equation in $\mathbb{R}^3$ of the surface?

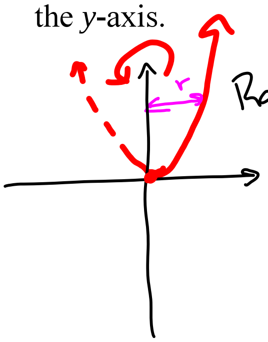
To find the equation of a surface of revolution, we need a radius function. The radius function will be a function of one variable only. The equation will take one of these forms:

Revolved about the *x*-axis: $y^2 + z^2 = [r(x)]^2$

Revolved about the *y*-axis: $x^2 + z^2 = [r(y)]^2$

Revolved about the *z*-axis: $x^2 + y^2 = [r(z)]^2$

Example 14: Find an equation for the surface generated by revolving the graph of $y = x^2$ about the *y*-axis.

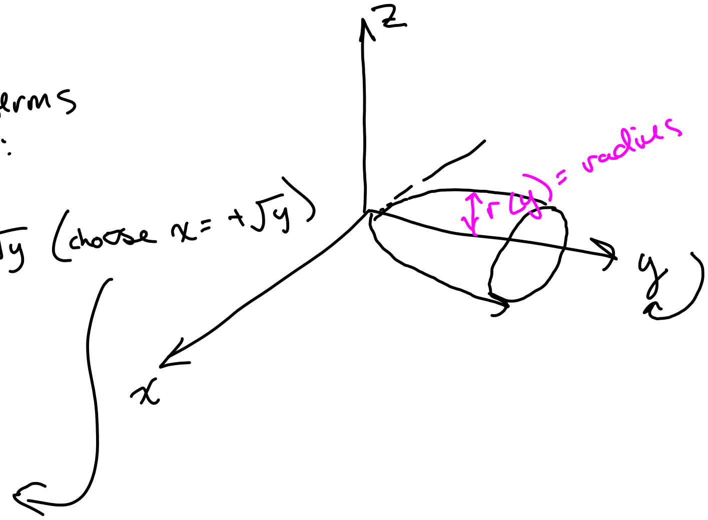


Radius $= r = x$

write x in terms of y:

$y = x^2$

$x = \pm\sqrt{y}$ (choose $x = +\sqrt{y}$)



Equation will take the form

$x^2 + z^2 = [r(y)]^2$

$x^2 + z^2 = (\sqrt{y})^2$

$\boxed{x^2 + z^2 = y}$

Example 15: Find a curve that will generate the surface $x^2 + z^2 - y^2 = 1$. Specify the axis of revolution.

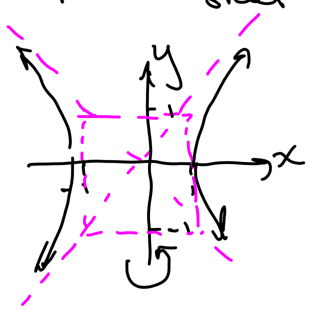
Hyperboloid of 1 sheet

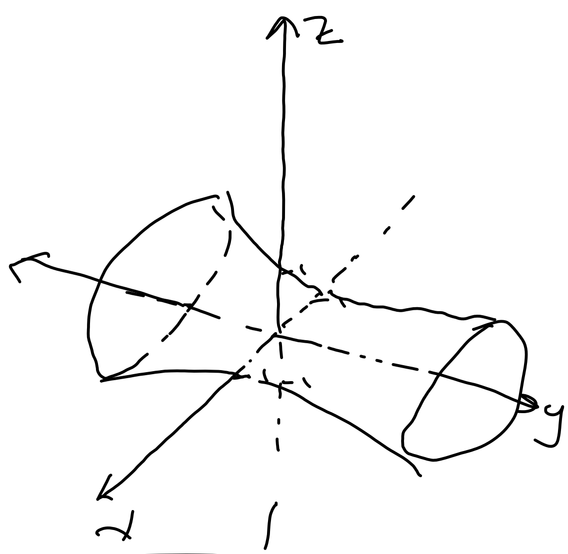
Traces:

xy: ($z=0$) $x^2 - y^2 = 1$ hyperbola

yz: ($x=0$): $z^2 - y^2 = 1$ hyperbola

xz: ($y=0$) $x^2 + z^2 = 1$ circle





Choose right-hand branch: ($x > 0$)

To generate the curve, you would revolve the graph of $x^2 - y^2 = 1$ around the y-axis (really just the right half $x = \sqrt{1+y^2}$)

Check: Does $x = \sqrt{1+y^2}$ give us the right equation?

$x^2 + z^2 = [r(y)]^2$

$x^2 + z^2 = (\sqrt{1+y^2})^2$

$x^2 + z^2 = 1 + y^2$

$x^2 + z^2 - y^2 = 1$ ✓

Note: Could start with $x^2 + z^2 - y^2 = 1$ and write as

$x^2 + z^2 = [r(y)]^2$

$x^2 + z^2 = 1 + y^2$.

So, $r(y) = \sqrt{1+y^2}$

$r^2 = 1 + y^2$

Choose either $x^2 = 1 + y^2$ or $z^2 = 1 + y^2$