12.1: Vector-Valued Functions

In $\mathbb{R}^2$, a *plane curve* can be described using parametric equations $x = f(t)$ and $y = g(t)$, where f and g are continuous functions of t on an interval I. (See Section 10.2 for a review.)

In $\mathbb{R}^3$, a *space curve* can be described using parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$, where f, g, and h are continuous functions of t on an interval I.

The plane curve or space curve is defined to be the *graph* (the set of ordered pairs or ordered triples) along with the defining parametric equations.

Note: The same graph can be generated by different sets of parametric equations. These are considered different curves, even though their graphs are the same.

Example 1: Sketch the graph defined by $x(t) = t^2 - 2$ and $y(t) = 1 + t$.

Example 2: Compare the graph defined by $x(t) = 2\sin t$ and $y(t) = 3\cos t$ with the graph defined by $x(t) = 2\cos 2t$ and $y(t) = 3\sin 2t$.

Real-valued functions: Output is a real number.
Vector-valued functions: Output is a vector.

<u>Definition</u>: A function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} = \langle f(t), g(t) \rangle \qquad \text{(in } \mathbb{R}^2\text{)}$$

or

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \langle f(t), g(t), h(t) \rangle \qquad \text{(in } \mathbb{R}^3\text{)}$$

is called a *vector-valued function*. The component functions *f*, *g*, and *h* are real-valued functions of the parameter *t*.

When working with vector-valued functions, we usually think of each output as a *position vector*. The initial point of a position vector is the origin; the terminal point is a point on a curve. Thus, as *t* increases, the vector-valued function traces the graph of a curve.

<u>Example 3:</u> Sketch the curve represented by $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$.

<u>Example 4:</u> What curve is represented by $\mathbf{r}(t) = \langle 3 - t, 4 + 2t, 6 - 3t \rangle$?

Example 5: Sketch the curve represented by $\mathbf{r}(t) = (2t-1)\,\mathbf{i} + (t^2+1)\,\mathbf{j} + (4-t^2)\,\mathbf{k}$.

Domain of vector-valued functions:

For a value of t to be in the domain of a vector-valued function, it needs to be in the domain of all the component functions.

Example 6: Find the domain of $\mathbf{s}(t) = \left\langle \ln(t+1), \sin(t), \dfrac{1}{t-2} \right\rangle$.

Example 7: Find the domain of $\mathbf{s}(t) = \left\langle \sqrt{4-1}, e^t, \dfrac{3}{t} \right\rangle$.

Representing a curve by a vector-valued function:

__**Example 8:**__ Represent the plane curve $y = 4 - x^2$ by a vector-valued function.

__**Example 9:**__ Represent the plane curve $x^2 + \frac{y^2}{5} = 1$ by a vector-valued function.

__**Example 10:**__ Find a vector-valued function that represents the line segment joining points $P(3,4,-2)$ and $Q(-4,-3,-1)$.

__**Example 11:**__ Find a vector-valued function that represents the curve of intersection of the paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.

Limits:

Definition: Limit of a Vector-Valued Function

If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, then $\lim\limits_{x\to a} \mathbf{r}(t) = \left\langle \lim\limits_{x\to a} f(t), \lim\limits_{x\to a} g(t) \right\rangle$, provided these limits of the component functions exist.

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\lim\limits_{x\to a} \mathbf{r}(t) = \left\langle \lim\limits_{x\to a} f(t), \lim\limits_{x\to a} g(t), \lim\limits_{x\to a} h(t) \right\rangle$, provided these limits of the component functions exist.

The properties of limits (sum, difference, scalar multiples, etc.) for vector-valued functions are similar to those for real-valued functions.

Note: The limit of a vector-valued function is a vector.

Example 12: Suppose that $\mathbf{r}(t) = \dfrac{\sin t}{t}\,\mathbf{i} + \dfrac{t^2 - 1}{2t^2 + 1}\,\mathbf{j} + \cos(t)\,\mathbf{k}$. Find $\lim\limits_{x\to 0} \mathbf{r}(t)$.

Example 13: Determine $\lim\limits_{x\to 0} \left\langle \dfrac{e^t - 1}{t}, \dfrac{\sqrt{1+t} - 1}{t}, \dfrac{3}{1+t} \right\rangle$.

Continuity:

> <u>Definition</u>:
>
> A vector-valued function **r** is *continuous at a point* given by $t = a$ if $\lim\limits_{x \to a} \mathbf{r}(t)$ exists and $\lim\limits_{x \to a} \mathbf{r}(t) = \mathbf{r}(a)$.
>
> A vector-valued function **r** is *continuous on an interval I* if it is continuous at every point in the interval.

<u>Example 14:</u> On what intervals is $\mathbf{r}(t) = \left\langle \tan(t), t, t^2 \right\rangle$ continuous?

<u>Example 15:</u> On what intervals is $\mathbf{r}(t) = \left\langle \sqrt{4 - t^2}, \frac{1}{t}, \frac{t-3}{5} \right\rangle$ continuous?