12.1: Vector-Valued Functions

In $\mathbb{R}^2$, a *plane curve* can be described using parametric equations $x = f(t)$ and $y = g(t)$, where *f* and *g* are continuous functions of *t* on an interval *I*. (See Section 10.2 for a review.)

In $\mathbb{R}^3$, a *space curve* can be described using parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$, where *f*, *g*, and *h* are continuous functions of *t* on an interval *I*.

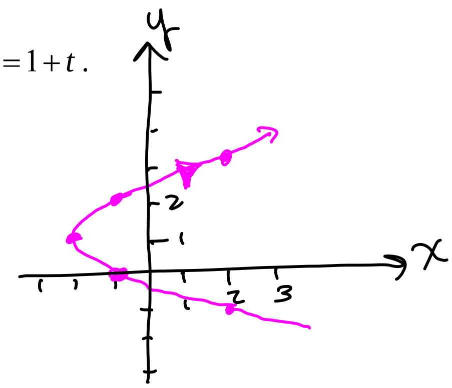
The plane curve or space curve is defined to be the *graph* (the set of ordered pairs or ordered triples) along with the defining parametric equations.

Note: The same graph can be generated by different sets of parametric equations. These are considered different curves, even though their graphs are the same.

Example 1: Sketch the graph defined by $x(t) = t^2 - 2$ and $y(t) = 1 + t$.

In $\mathbb{R}^2$

t	x	y	
-2	$(-2)^2-2=2$	$1-2=-1$	$(2,-1)$
-1	$(-1)^2-2=-1$	$1-1=0$	$(-1,0)$
0	$0^2-2=-2$	$1+0=1$	$(-2,1)$
1	$1^2-2=-1$	$1+1=2$	$(-1,2)$
2	$2^2-2=2$	$1+2=3$	$(2,3)$

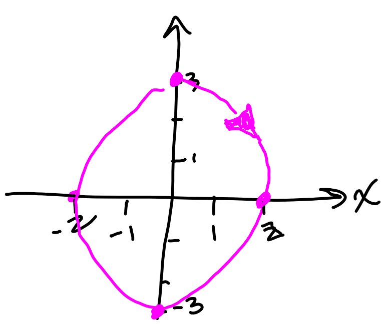


Example 2: Compare the graph defined by $x(t) = 2\sin t$ and $y(t) = 3\cos t$ with the graph defined by $x(t) = 2\cos 2t$ and $y(t) = 3\sin 2t$.

In $\mathbb{R}^2$

$x(t) = 2\sin t$
$y(t) = 3\cos t$

t	x	y
0	0	3
$\pi/2$	2	0
π	0	-3
$\frac{3\pi}{2}$	-2	0
2π	0	3

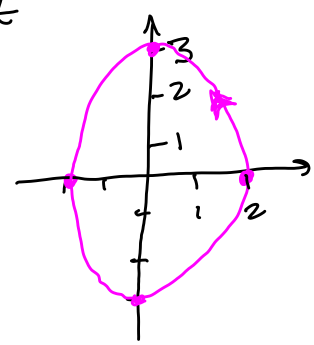


$0 \le t < 2\pi$ gives 1 revolution

clockwise

$x(t) = 2\cos 2t$
$y(t) = 3\sin 2t$

t	x	y
0	2	0
$\pi/4$	0	3
$\pi/2$	-2	0
$\frac{3\pi}{4}$	0	-3
π	2	0



$0 \le t < \pi$ gives 1 revolution

counterclockwise

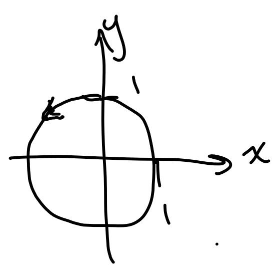
Real-valued functions: Output is a real number.
Vector-valued functions: Output is a vector.

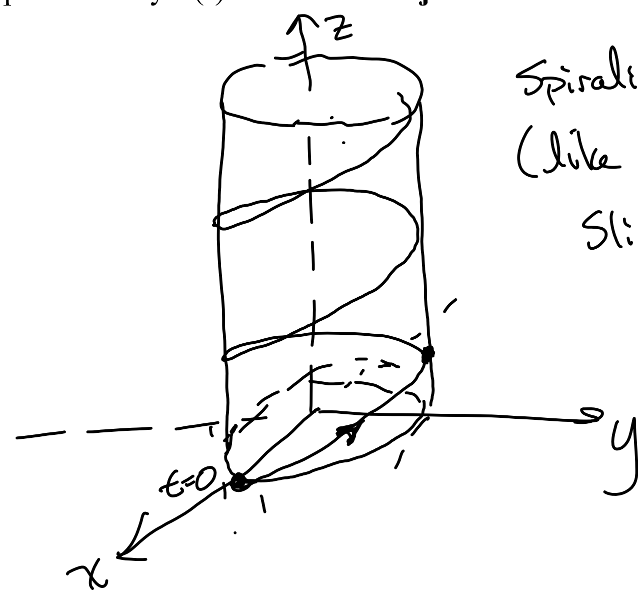
> Definition: A function of the form
>
> $$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} = \langle f(t), g(t) \rangle \qquad (\text{in } \mathbb{R}^2)$$
>
> or
>
> $$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \langle f(t), g(t), h(t) \rangle \qquad (\text{in } \mathbb{R}^3)$$
>
> is called a *vector-valued function*. The component functions *f*, *g*, and *h* are real-valued functions of the parameter *t*.

When working with vector-valued functions, we usually think of each output as a *position vector*. The initial point of a position vector is the origin; the terminal point is a point on a curve. Thus, as *t* increases, the vector-valued function traces the graph of a curve.

Example 3: Sketch the curve represented by $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$.

Projection onto xy plane:





Spiraling upward (like a Slinky)

Example 4: What curve is represented by $\mathbf{r}(t) = \langle 3-t, 4+2t, 6-3t \rangle$?

This is a line. It has direction vector $\langle -1, 2, -3 \rangle = \vec{v}$. It includes the point $(3,4,6)$.

<u>Example 5:</u> Sketch the curve represented by $\mathbf{r}(t) = (2t-1)\,\mathbf{i} + (t^2+1)\,\mathbf{j} + (4-t^2)\,\mathbf{k}$.

$\vec{r}(-3) = \langle -7, 10, -5 \rangle$

$\vec{r}(-2) = \langle -5, 5, 0 \rangle$

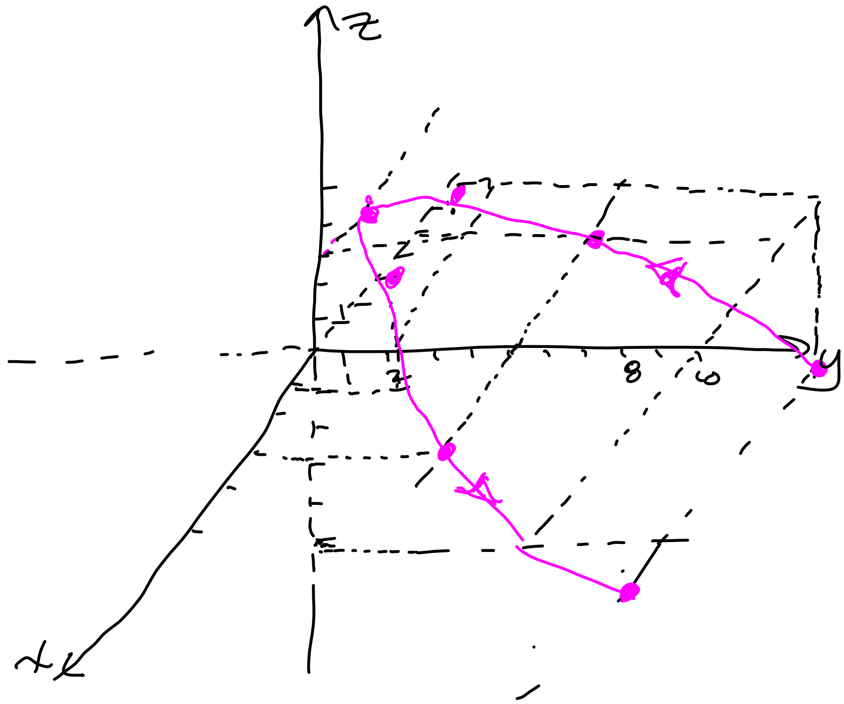
$\vec{r}(-1) = \langle -3, 2, 3 \rangle$

$\vec{r}(0) = \langle -1, 1, 4 \rangle$

$\vec{r}(1) = \langle 1, 2, 3 \rangle$

$\vec{r}(2) = \langle 3, 5, 0 \rangle$

$\vec{r}(3) = \langle 5, 10, -5 \rangle$



Domain of vector-valued functions:

For a value of t to be in the domain of a vector-valued function, it needs to be in the domain of <u>all</u> the component functions.

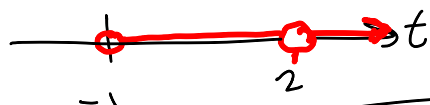
<u>Example 6:</u> Find the domain of $\mathbf{s}(t) = \left\langle \ln(t+1), \sin(t), \dfrac{1}{t-2} \right\rangle$.

$\ln(t+1) \Rightarrow t+1 > 0$

$t > -1$

$\sin(t)$ Domain $\mathbb{R}$

$\dfrac{1}{t-2} \Rightarrow t \neq 2$



Domain of $\vec{s}$: $\boxed{(-1,2) \cup (2,\infty)}$

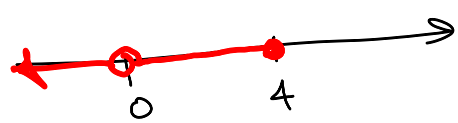
<u>Example 7:</u> Find the domain of $\mathbf{s}(t) = \left\langle \sqrt{4-t}, e^t, \dfrac{3}{t} \right\rangle$.

$\sqrt{4-t} \Rightarrow 4-t \geq 0$

$4 \geq t$

$t \leq 4$

$\dfrac{3}{t} \Rightarrow t \neq 0$



Domain: $(-\infty, 0) \cup (0, 4]$

Representing a curve by a vector-valued function:

Example 8: Represent the plane curve $y = 4 - x^2$ by a vector-valued function.

Introduce a parameter t:

Let $x = t$

Then $y = 4 - t^2$

$$\boxed{\vec{r}(t) = \langle t, 4-t^2 \rangle}$$

or $\vec{r}(t) = t\,\hat{i} + (4-t^2)\,\hat{j}$

Example 9: Represent the plane curve $x^2 + \dfrac{y^2}{5} = 1$ by a vector-valued function.

For an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, you can let $x = a\cos t$, $y = b\sin t$

Here, let $x = 1\cos t$, $y = \sqrt{5}\sin t$

$$\boxed{\vec{r}(t) = \langle \cos t, \sqrt{5}\sin t \rangle}$$

Example 10: Find a vector-valued function that represents the line segment joining points $P(3,4,-2)$ and $Q(-4,-3,-1)$.

$\overrightarrow{PQ} = \langle -7, -7, 1 \rangle$

$$\left.\begin{aligned} x &= 3-7t \\ y &= 4-7t \\ z &= -2+t \end{aligned}\right\} \text{starting with } P.$$

Note: $t=0 \Rightarrow P(3,4,-2)$

$t = 1 \Rightarrow Q(-4,-3,-1)$

$$\boxed{\vec{r}(t) = \langle 3-7t, 4-7t, -2+t \rangle}$$

Example 11: Find a vector-valued function that represents the curve of intersection of the paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.

Let $x = t$.

Then $y = x^2 \Rightarrow y = t^2$

$z = 4x^2 + y^2 \Rightarrow z = 4t^2 + (t^2)^2$

$z = 4t^2 + t^4$

$$\boxed{\vec{r}(t) = \langle t, t^2, 4t^2+t^4 \rangle}$$

Limits:

<u>Definition: Limit of a Vector-Valued Function</u>

If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, then $\lim_{x\to a} \mathbf{r}(t) = \left\langle \lim_{x\to a} f(t), \lim_{x\to a} g(t) \right\rangle$, provided these limits of the component functions exist.

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\lim_{x\to a} \mathbf{r}(t) = \left\langle \lim_{x\to a} f(t), \lim_{x\to a} g(t), \lim_{x\to a} h(t) \right\rangle$, provided these limits of the component functions exist.

The properties of limits (sum, difference, scalar multiples, etc.) for vector-valued functions are similar to those for real-valued functions.

<u>Note</u>: The limit of a vector-valued function is a <u>vector</u>.

$\lim_{t\to 0} \vec{r}(t)$

<u>Example 12:</u> Suppose that $\mathbf{r}(t) = \frac{\sin t}{t}\mathbf{i} + \frac{t^2-1}{2t^2+1}\mathbf{j} + \cos(t)\,\mathbf{k}$. Find $\lim_{t\to 0} \mathbf{r}(t)$.

$$\lim_{t\to 0} \vec{r}(t) = \left\langle 1, \frac{-1}{1}, 1 \right\rangle$$

$$= \boxed{\langle 1, -1, 1 \rangle}$$

<u>Example 13:</u> Determine $\lim_{t\to 0} \left\langle \frac{e^t - 1}{t}, \frac{\sqrt{1+t}-1}{t}, \frac{3}{1+t} \right\rangle$.

$$\lim_{t\to 0} \left\langle \frac{e^t-1}{t}, \frac{\sqrt{1+t}-1}{t}, \frac{3}{1+t} \right\rangle$$

$$= \lim_{t\to 0} \left\langle \frac{e^t}{1}, \frac{\frac{1}{2}(1+t)^{-1/2}}{1}, 3 \right\rangle \quad [\text{L'Hospital}]$$

$$= \left\langle 1, \frac{1}{2\sqrt{1+0}}, 3 \right\rangle = \boxed{\left\langle 1, \frac{1}{2}, 3 \right\rangle}$$

Continuity:

> Definition:
>
> A vector-valued function **r** is *continuous at a point* given by $t = a$ if $\lim_{x \to a} \mathbf{r}(t)$ exists and $\lim_{x \to a} \mathbf{r}(t) = \mathbf{r}(a)$.
>
> A vector-valued function **r** is *continuous on an interval I* if it is continuous at every point in the interval.

Example 14: On what intervals is $\mathbf{r}(t) = \langle \tan(t), t, t^2 \rangle$ continuous?

$\vec{r}$ is continuous except for

$t = \frac{(2k+1)\pi}{2}$, k any integer.

Example 15: On what intervals is $\mathbf{r}(t) = \left\langle \sqrt{4 - t^2}, \frac{1}{t}, \frac{t-3}{5} \right\rangle$ continuous?

$\sqrt{4-t^2}$ is defined for $-2 \le t \le 2$.

$t \neq 0$ [because of $\frac{1}{t}$]

$\vec{r}$ is continuous on $[-2, 0) \cup (0, 2]$