12.2: Differentiation and Integration of Vector-Valued Functions

Differentiation:

Definition: The derivative of a vector-valued function $\mathbf{r}$ is

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(t)$ exists, then r is differentiable at t. If $\mathbf{r}$ is differentiable for all t in an open interval I, then $\mathbf{r}$ is differentiable on the interval I.

Theorem:

If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, where f and g are differentiable functions of t, then $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$.

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g, and h are differentiable functions of t, then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

Example 1: Suppose $\mathbf{r}(t) = 4\sqrt{t}\,\mathbf{i} + t^2\sqrt{t}\,\mathbf{j} + \ln(t^2)\,\mathbf{k}$. Find $\mathbf{r}'(t)$.

Example 2: Suppose $\mathbf{r}(t) = \langle 7\cos t, 4\sin 2t \rangle$. Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t)\cdot\mathbf{r}''(t)$.

Definition:

The parametrization of the curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is considered *smooth* on an open interval I if

1. the component derivatives f', g', and h' are continuous on I

 and

2. $\mathbf{r}'(t)$ is nonzero on all of I.

Example 3: Suppose $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (t^2 - 1)\,\mathbf{j} + \frac{1}{4}t\,\mathbf{k}$, $t \geq 0$. Find the open intervals on which the curve generated by the function is smooth.

Example 4: Suppose $\mathbf{r}(t) = \langle 5\cos t - \cos 5t, 5\sin t - \sin 5t \rangle$, $0 \le t \le 2\pi$. Find the open intervals on which the curve generated by the function is smooth.

Properties of the Derivative:

1. $\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

2. $\frac{d}{dt}[\mathbf{r}(t) + \mathbf{u}(t)] = \mathbf{r}'(t) + \mathbf{u}'(t)$

3. $\frac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$

4. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

5. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

6. $\frac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$

7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$

Integration:

Definition:

Suppose $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and that f, g, and h are continuous on $[a,b]$. Then,

$$\int \mathbf{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$$

and

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle .$$

Note: The constants of integration for the components can be combined into a vector **c**.

Example 5: Calculate $\int \left(t^2\,\mathbf{i} + \cos t\,\mathbf{j} + 5\,\mathbf{k}\right) dt$.

Example 6: Calculate $\int_1^2 \left(\ln t\,\mathbf{i} + \frac{1}{t+2}\,\mathbf{j} + (t-1)^2\,\mathbf{k} \right) dt$.

Example 7: Suppose $\mathbf{r}'(t) = \left\langle e^{3t}, \sin(4t), 4t^3 \right\rangle$ and $\mathbf{r}(0) = \langle 3, -2, 1 \rangle$. Find $\mathbf{r}(t)$.

Example 8: Suppose $\mathbf{r}''(t) = -4\cos(t)\,\mathbf{j} - 3\sin(t)\,\mathbf{k}$, $\mathbf{r}'(0) = 2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{r}(0) = 4\,\mathbf{j}$. Find $\mathbf{r}(t)$.