12.2: Differentiation and Integration of Vector-Valued Functions

Differentiation:

Definition: The derivative of a vector-valued function **r** is

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(t)$ exists, then r is differentiable at t. If **r** is differentiable for all t in an open interval I, then **r** is differentiable on the interval I.

$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$

Theorem:

If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, where f and g are differentiable functions of t, then $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$.

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g, and h are differentiable functions of t, then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

Example 1: Suppose $\mathbf{r}(t) = 4\sqrt{t}\ \mathbf{i} + t^2\sqrt{t}\ \mathbf{j} + \ln(t^2)\ \mathbf{k}$. Find $\mathbf{r}'(t)$.

$\vec{r}(t) = \langle 4t^{1/2}, t^{5/2}, \ln(t^2) \rangle$

$\vec{r}'(t) = \langle 4(\frac{1}{2}t^{-1/2}), \frac{5}{2}t^{3/2}, \frac{1}{t^2}(2t) \rangle$

$= \boxed{\langle \frac{2}{\sqrt{t}}, \frac{5}{2}t^{3/2}, \frac{2}{t} \rangle}$

Example 2: Suppose $\mathbf{r}(t) = \langle 7\cos t, 4\sin 2t \rangle$. Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

$\vec{r}'(t) = \langle -7\sin t, 4(\cos(2t))(2) \rangle = \langle -7\sin t, 8\cos(2t) \rangle$

$\vec{r}''(t) = \langle -7\cos t, -8(\sin(2t))(2) \rangle = \langle -7\cos t, -16\sin(2t) \rangle$

$\vec{r}'(t) \cdot \vec{r}''(t) = (-7\sin t)(-7\cos t) + (8\cos 2t)(-16\sin 2t)$

$= \boxed{49\sin t\cos t - 128\sin 2t\cos 2t}$

> Definition:
>
> The parametrization of the curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is considered *smooth* on an open interval I if
>
> 1. the component derivatives f', g', and h' are continuous on I
>
> and
>
> 2. $\mathbf{r}'(t)$ is nonzero on all of I.

Example 3: Suppose $\mathbf{r}(t) = \sqrt{t}\ \mathbf{i} + (t^2 - 1)\ \mathbf{j} + \frac{1}{4}t\ \mathbf{k}$, $t \geq 0$. Find the open intervals on which the curve generated by the function is smooth.

$\vec{r}(t) = \langle t^{1/2}, t^2 - 1, \frac{1}{4}t \rangle$

$\vec{r}'(t) = \langle \frac{1}{2}t^{-1/2}, 2t, \frac{1}{4} \rangle = \langle \frac{1}{2\sqrt{t}}, 2t, \frac{1}{4} \rangle$

$\frac{1}{2\sqrt{t}}$ only defined for $t > 0$.

$\boxed{\vec{r} \text{ is smooth for } t > 0}$

Example 4: Suppose $\mathbf{r}(t)=\langle 5\cos t-\cos 5t, 5\sin t-\sin 5t\rangle$, $0\le t\le 2\pi$. Find the open intervals on which the curve generated by the function is smooth.

$\vec{r}(t) = \langle 5\cos t - \cos 5t, 5\sin t - \sin 5t\rangle$

$\vec{r}\,'(t) = \langle -5\sin t - (-5\sin 5t), 5\cos t - 5\cos 5t\rangle$

$= \langle -5\sin t + 5\sin 5t, 5\cos t - 5\cos 5t\rangle$

$\vec{r}(t) = \vec{0}$ for $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

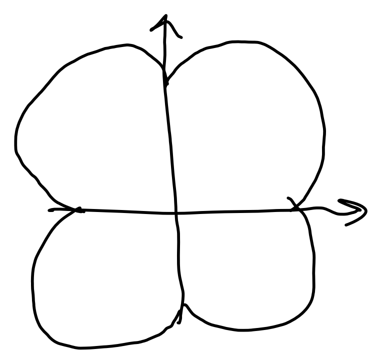
$\vec{r}(0) = \langle -5\sin 0 + 5\sin 0, 5\cos 0 - 5\cos 0\rangle = \langle 0, 5-5\rangle = \langle 0,0\rangle$

$\vec{r}(\frac{\pi}{2}) = \langle -5 + 5\sin\frac{5\pi}{2}, 0-0\rangle = \langle -5+5, 0\rangle = \langle 0,0\rangle$

$\vec{r}(\pi) = \langle 0, -5 - 5\cos 5\pi\rangle = \langle 0, -5+5\rangle = \langle 0,0\rangle$

$\vec{r}(\frac{3\pi}{2}) = \langle -5(-1) + 5\sin\frac{15\pi}{2}, 0\rangle = \langle 5-5, 0\rangle = \langle 0,0\rangle$

$\boxed{\text{Not smooth at } t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi}$



Properties of the Derivative:

1. $\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

2. $\frac{d}{dt}[\mathbf{r}(t)+\mathbf{u}(t)] = \mathbf{r}'(t)+\mathbf{u}'(t)$

Product Rule

3. $\frac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t)+w'(t)\mathbf{r}(t)$

4. $\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)] = \mathbf{r}(t)\cdot\mathbf{u}'(t)+\mathbf{r}'(t)\cdot\mathbf{u}(t)$

5. $\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{u}(t)] = \mathbf{r}(t)\times\mathbf{u}'(t)+\mathbf{r}'(t)\times\mathbf{u}(t)$

Chain Rule →

6. $\frac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$

7. If $\mathbf{r}(t)\cdot\mathbf{r}(t)=c$, then $\mathbf{r}(t)\cdot\mathbf{r}'(t)=0$

Prop #7: Why is it true?

Suppose $\vec{r}(t)\cdot\vec{r}(t) = c$.

Let's calculate $\vec{r}(t)\cdot\vec{r}\,'(t)$.

Prop #4 $\Rightarrow$ $\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)] = \vec{r}(t)\cdot\vec{r}\,'(t) + \vec{r}\,'(t)\cdot\vec{r}(t) = 2\vec{r}(t)\cdot\vec{r}\,'(t)$

$\frac{1}{2}\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)] = \vec{r}(t)\cdot\vec{r}\,'(t)$

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$\hookrightarrow \frac{1}{2}\frac{d}{dt}[C] = \vec{r}(t)\cdot\vec{r}\,'(t)$

$\frac{1}{2}(0) = \vec{r}(t)\cdot\vec{r}\,'(t)$

$0 = \vec{r}(t)\cdot\vec{r}\,'(t)$. This means that

Integration: if $\underbrace{\vec{r}(t)\cdot\vec{r}(t)}_{\|\vec{r}(t)\|^2}$ is constant, then $\vec{r}(t)\perp\vec{r}\,'(t)$

Definition:

Suppose $\mathbf{r}(t)=\langle f(t), g(t), h(t)\rangle$ and that f, g, and h are continuous on $[a,b]$. Then,

$\int \mathbf{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle$

and

$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$.

Note: The constants of integration for the components can be combined into a vector **c**.

Example 5: Calculate $\int \left(t^2\,\mathbf{i}+\cos t\,\mathbf{j}+5\,\mathbf{k}\right) dt$.

$$\int \langle t^2, \cos t, 5\rangle\, dt = \left\langle \frac{t^3}{3}+C_1, \sin t + C_2, 5t + C_3\right\rangle$$
$$= \left\langle \frac{t^3}{3}, \sin t, 5t\right\rangle + \langle C_1, C_2, C_3\rangle$$
$$= \left\langle \frac{t^3}{3}, \sin t, 5t\right\rangle + \vec{c}$$
$$= \boxed{\frac{t^3}{3}\vec{i} + \sin t\,\vec{j} + 5t\vec{k} + \vec{c}}$$

Example 6: Calculate $\int_1^2 \left(\ln t\,\mathbf{i} + \frac{1}{t+2}\,\mathbf{j} + (t-1)^2\,\mathbf{k}\right) dt$.

$$\int_1^2 \left\langle \ln t, \frac{1}{t+2}, (t-1)^2\right\rangle dt$$
$$= \left\langle \int_1^2 \ln t\,dt, \int_1^2 \frac{1}{t+2}\,dt, \int_1^2 (t-1)^2\,dt\right\rangle$$
$$= \left\langle (t\ln t - t)\Big|_1^2, \ln|t+2|\Big|_1^2, \frac{(t-1)^3}{3}\Big|_1^2\right\rangle$$
$$= \left\langle 2\ln 2 - 2 - (1\ln 1 - 1), \ln 4 - \ln 3, \frac{1^3}{3} - \frac{0^3}{3}\right\rangle$$
$$= \left\langle 2\ln 2 - 2 - 0 + 1, \ln\left(\frac{4}{3}\right), \frac{1}{3}\right\rangle$$
$$= \boxed{\left\langle 2\ln 2 - 1, \ln\frac{4}{3}, \frac{1}{3}\right\rangle}$$

$\int \ln t\,dt$ $\quad u = \ln t \quad dv = dt$
$\quad du = \frac{1}{t}dt \quad v = t$

$$\int \ln t\,dt = uv - \int v\,du$$
$$= t\ln t - \int t\left(\frac{1}{t}\right)dt$$
$$= t\ln t - \int 1\,dt$$
$$= t\ln t - t + c$$

Previous example: could also write:

$$\left\langle t\ln t - t,\ \ln|t+2|,\ \frac{(t-1)^3}{3} \right\rangle\Big|_1^2$$

$$= \left\langle 2\ln 2 - 2, \ln 4, \frac{1}{3}\right\rangle - \left\langle 1\ln 1 - 1, \ln 3, \frac{0^3}{3}\right\rangle =$$

$$= \left\langle 2\ln 2 - 2, \ln 4, \frac{1}{3}\right\rangle - \langle -1, \ln 3, 0\rangle$$

$$= \boxed{\left\langle 2\ln 2 - 1, \ln\frac{4}{3}, \frac{1}{3}\right\rangle}$$

Example 7: Suppose $\mathbf{r}'(t) = \langle e^{3t}, \sin(4t), 4t^3\rangle$ and $\mathbf{r}(0) = \langle 3, -2, 1\rangle$. Find $\mathbf{r}(t)$.

This is a differential eqn with an initial condition.

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt$$

$$= \int \langle e^{3t}, \sin(4t), 4t^3\rangle\,dt$$

$$= \left\langle \frac{1}{3}e^{3t}, -\frac{1}{4}\cos(4t), \frac{4t^4}{4}\right\rangle + \vec{C} = \left\langle \frac{1}{3}e^{3t}, -\frac{1}{4}\cos(4t), t^4\right\rangle$$

$$\vec{r}(0) = \langle 3,-2,1\rangle \Rightarrow \vec{r}(0) = \left\langle \frac{1}{3}e^{3(0)}, -\frac{1}{4}\cos(4\cdot 0), 0^4\right\rangle + \vec{C} = \langle 3,-2,1\rangle$$

$$\left\langle \frac{1}{3}(1), -\frac{1}{4}(1), 0\right\rangle + \vec{C} = \langle 3,-2,1\rangle$$

$$\vec{C} = \langle 3,-2,1\rangle - \left\langle \frac{1}{3}, -\frac{1}{4}, 0\right\rangle = \left\langle \frac{8}{3}, -\frac{7}{4}, 1\right\rangle$$

$$\boxed{\vec{r}(t) = \left\langle \frac{1}{3}e^{3t} + \frac{8}{3}, -\frac{1}{4}\cos 4t - \frac{7}{4}, t^4 + 1\right\rangle}$$

Example 8: Suppose $\mathbf{r}''(t) = -4\cos(t)\,\mathbf{j} - 3\sin(t)\,\mathbf{k}$, $\mathbf{r}'(0) = 2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{r}(0) = 4\,\mathbf{j}$. Find $\mathbf{r}(t)$.

$$\vec{r}\,''(t) = \langle 0, -4\cos t, -3\sin t\rangle$$

$$\vec{r}\,'(t) = \int \vec{r}\,''(t)\,dt = \int \langle 0, -4\cos t, -3\sin t\rangle\,dt = \langle 0, -4\sin t, 3\cos t\rangle + \vec{C}_1$$

$$\vec{r}\,'(0) = \langle 0, -4\sin 0, 3\cos 0\rangle + \vec{C}_1 = \langle 2, 0, 3\rangle$$

$$\vec{C}_1 = \langle 2,0,3\rangle - \langle 0,0,3\rangle = \langle 2,0,0\rangle$$

$$\vec{r}\,'(t) = \langle 0, -4\sin t, 3\cos t\rangle + \langle 2,0,0\rangle = \langle 2, -4\sin t, 3\cos t\rangle$$

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \int \langle 2, -4\sin t, 3\cos t\rangle = \langle 2t, 4\cos t, 3\sin t\rangle + \vec{C}_2$$

$$\vec{r}(0) = \langle 2(0), 4\cos 0, 3\sin 0\rangle + \vec{C}_2 = \langle 0,4,0\rangle$$

$$\langle 0,4,0\rangle + \vec{C}_2 = \langle 0,4,0\rangle$$

$$\vec{C}_2 = \langle 0,4,0\rangle - \langle 0,4,0\rangle = \langle 0,0,0\rangle = \vec{0}$$

$$\boxed{\vec{r}(t) = \langle 2t, 4\cos t, 3\sin t\rangle}$$

Homework Qs

11.7 #83 $x^2+y^2+z^2=25$

b) Spherical:

$$\rho=\sqrt{x^2+y^2+z^2}$$

$$\rho^2=x^2+y^2+z^2$$

$$\rho^2=25$$

$$\boxed{\rho=5}$$

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11.7 #85

$$x^2+y^2+z^2-2z=0$$

Complete the square in z

$$\underbrace{x^2+y^2}_{r^2}+(z-1)^2=1$$

$$r^2=\rho^2\sin^2\phi$$

$$z=\rho\cos\phi$$

$$\underbrace{x^2+y^2+z^2}_{\rho^2}-2\underbrace{z}_{\rho\cos\phi}=0$$

$$\rho^2-2\rho\cos\phi=0$$

Divide by ρ:

$$\rho-2\cos\phi=0$$

$$\rho=2\cos\phi$$