12.3: Velocity and Acceleration

Suppose an object's position is given by the vector-valued function $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$. Then the first and second derivatives represent the velocity and acceleration. Note that the velocity and acceleration are vectors that have both magnitude and direction.

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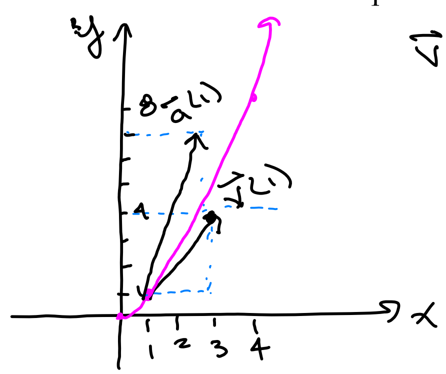
> Definition: Suppose x, y, and z are twice-differentiable functions of t, and $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. Then the velocity vector, acceleration vector, and speed at time t are:
>
> Velocity = $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
>
> Acceleration = $a(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$.
>
> Speed = $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$.
>
> (This applies to vectors in $\mathbb{R}^2$ also.)

Note: The velocity vector is also known as the tangent vector. When its initial point is placed at the point on the curve, it is tangent to the curve.

Example 1: Suppose an object's position is given by the vector $\mathbf{r}(t) = \langle t^2, t^3 \rangle$. Sketch the path of the object, and the velocity and acceleration vectors at the point $(1,1)$.

t	$x(t)=t^2$	$y(t)=t^3$
0	$0^2=0$	$0^3=0$ $(0,0)$
1	$1^2=1$	$1^3=1$ $(1,1)$
2	$2^2=4$	$2^3=8$ $(4,8)$
3	$3^2=9$	$3^3=27$ $(9,27)$



$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, 3t^2 \rangle$

$\vec{v}(1) = \langle 2(1), 3(1)^2 \rangle$

$= \langle 2, 3 \rangle$

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

$= \langle 2, 6t \rangle$

$\vec{a}(1) = \langle 2, 6(1) \rangle = \langle 2, 6 \rangle$

Example 2: Suppose an object's position is given by the vector $\mathbf{r}(t) = \langle t^2, t, t^{3/2} \rangle$. Find the velocity vector, the acceleration vector, and the speed in terms of t.

$$\vec{v}(t) = \vec{r}\,'(t) = \boxed{\langle 2t, 1, \tfrac{3}{2} t^{1/2} \rangle}$$

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \boxed{\langle 2, 0, \tfrac{3}{4} t^{-1/2} \rangle}$$

Speed:

$$\|\vec{v}\,'(t)\| = \sqrt{(2t)^2 + (1)^2 + (\tfrac{3}{2} t^{1/2})^2}$$

$$= \boxed{\sqrt{4t^2 + 1 + \tfrac{9}{4} t}}$$

Projectile motion:

$\vec{v}_0 =$
$= \langle v_0\cos\theta, v_0\sin\theta \rangle$

> Theorem:
> If a projectile is launched from an initial height h with an initial speed v_0 and angle of elevation θ, and if air resistance is neglected, then the projectile's path is described by the vector function
>
> $$\mathbf{r}(t) = \left\langle (v_0\cos\theta)t, h + (v_0\sin\theta)t - \frac{1}{2}gt^2 \right\rangle,$$
>
> where g is the acceleration of gravity.

$\vec{a}(t) = \langle 0, -g \rangle$

$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 0, -g \rangle\,dt$

$= \langle 0, -gt \rangle + \vec{c}_1$

$\vec{v}(t) = \langle 0, -gt \rangle + \vec{c}_1$

$\vec{v}_0(0) = \langle v_0\cos\theta, v_0\sin\theta \rangle$

$\Rightarrow \vec{v}_0 = \langle 0, -g(0) \rangle + \vec{c}_1 = \langle v_0\cos\theta, v_0\sin\theta \rangle$

$\vec{c}_1 = \langle v_0\cos\theta, v_0\sin\theta \rangle$

$\vec{v}(t) = \langle v_0\cos\theta, v_0\sin\theta - gt \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle v_0\cos\theta, v_0\sin\theta - gt \rangle\,dt$

$= \langle (v_0\cos\theta)t, (v_0\sin\theta)t - g\frac{t^2}{2} \rangle + \vec{c}_2$

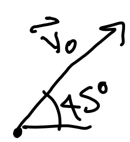
$\vec{r}(0) = \langle 0, h \rangle$

$\vec{r}(0) = \langle 0, 0 \rangle + \vec{c}_2 = \langle 0, h \rangle$

$\vec{c}_2 = \langle 0, h \rangle$

$\vec{r}(t) = \langle (v_0\cos\theta)t, h + (v_0\sin\theta)t - \frac{1}{2}gt^2 \rangle$

Example 3: Suppose a projectile is fired at a height of 3000 feet above the ground with an initial velocity of 900 feet/second at an angle of 45 degrees above the horizontal. Determine the maximum height and range of the projectile.



$\vec{r}(t) = \langle (v_0\cos\theta)t, h + (v_0\sin\theta)t - \frac{1}{2}gt^2 \rangle$

$= \langle (900\,\text{ft/sec}\cos 45^\circ)t, 3000\,\text{ft} + (900\,\frac{\text{ft}}{\text{sec}}\sin 45^\circ)t - \frac{1}{2}(32\frac{\text{ft}}{\text{s}^2})t^2 \rangle$

acceleration of gravity:
$g = 32\,\text{ft/s}^2$

$x(t) = (v_0\cos\theta)t = (900\,\frac{\text{ft}}{\text{sec}}\cos 45^\circ)t = 636.396\,\text{ft/sec}\;t$

$y(t) = 3000\,\text{ft} + (900\,\frac{\text{ft}}{\text{sec}}\sin 45^\circ)t - 16\,\frac{\text{ft}}{\text{sec}^2}t^2$

$= 3000\,\text{ft} + (636.396\,\text{ft/sec})t - (16\,\text{ft/sec}^2)t^2$

Max height occurs when $y'(t) = 0$: $y'(t) = 636.396\,\frac{\text{ft}}{\text{sec}} - 32\,\frac{\text{ft}}{\text{s}^2}\,t = 0$

$636.396\,\text{ft/sec} = 32\,\frac{\text{ft}}{\text{sec}^2}\,t$

$\frac{636.396}{32} \cdot \frac{\text{ft}}{\text{sec}}\left(\frac{\text{sec}^2}{\text{ft}}\right) = t$

$19.887\,\text{sec} = t$

(at max height)

See next page

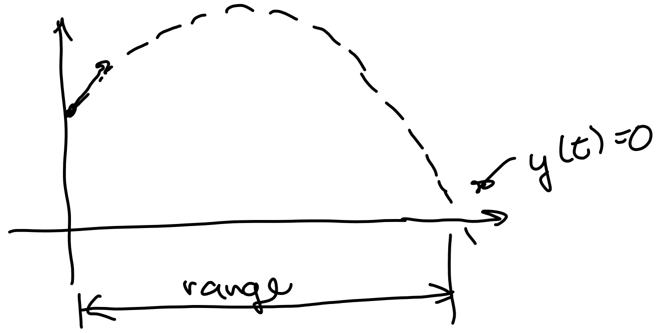
Ex 4 cont'd:

Max height is $y(19.887) = 3000\,ft + \left(636.396\,\frac{ft}{sec}\right)(19.887\,sec) - \frac{16\,ft}{sec^2}(19.887\,sec)^2$

$= 9328.125\,ft$

$\approx \boxed{9328\,ft}$

Find range:



Find t when $y(t) = 0$:

$0 = y(t) = 3000\,ft + (636.396\,ft/sec)t - (16\,ft/sec^2)t^2 = 0$

$= (-16\,ft/sec^2)t^2 + (636.396\,ft/sec)t + 3000\,ft$

Quadratic formula:

$$t = \frac{-636.396\,ft/sec \pm \sqrt{(636.396\,ft/s)^2 - 4(3000\,ft)(-16\,ft/sec^2)}}{2(-16\,ft/sec^2)}$$

$= 44.03\,sec$

Put this into $x(t)$ to find out how far it goes before hitting the ground.

$x(44.03\,sec) = (636.396\,ft/sec)(44.03\,sec)$

$\approx \boxed{28\,021\,ft}$

Example 4: Suppose a projectile is fired from ground level at an angle of 12° above the horizontal. Determine the minimum initial velocity necessary for the projectile to have a range of 200 feet.

$x(t) = (v_0 \cos 12°)t$

$y(t) = h + (v_0 \sin 12°)t - \frac{1}{2}gt^2$

$= 0 + (v_0 \sin 12°)t - \frac{1}{2}(32\,ft/sec^2)t^2$

$= (v_0 \sin 12°)t - (16\,ft/sec^2)t^2$

Let t_1 = time when it hits the ground

$x(t_1) = 200\,ft = v_0 \cos 12°\, t_1$

$t_1 = \frac{200\,ft}{v_0 \cos 12°}$

At time t_1, $y(t_1) = 0$. (hits ground)

$0 = (v_0 \sin 12°)t_1 - \left(\frac{16\,ft}{sec^2}\right)t_1^2$

Factor out t_1:

$0 = t_1\left(v_0 \sin 12° - \frac{16\,ft}{sec^2}t_1\right)$

$t_1 = 0$ or $v_0 \sin 12° - \frac{16\,ft}{sec^2}t_1 = 0$

Substitute $t_1 = \frac{200\,ft}{v_0 \cos 12°}$:

$v_0 \sin 12° - \frac{16\,ft}{sec^2}\left(\frac{200\,ft}{v_0 \cos 12°}\right) = 0$

Example 5: A baseball player at second base throws a ball 90 feet to the player at first base. The ball is released at 3 feet above the ground with an initial velocity of 70 miles per hour, at an angle of 15° above the horizontal. At what height does the first baseman catch the ball?

Ex 4 cont'd:

$v_0 \sin 12° = \frac{3200\,ft^2}{sec^2\, v_0 \cos 12°}$

multiply by $v_0 \cos 12°$:

$v_0^2 \sin 12° \cos 12° = 3200\,ft^2/sec^2$

$v_0^2 = \frac{3200\,ft^2/sec^2}{\sin 12° \cos 12°}$

$v_0 = \sqrt{\frac{3200\,ft^2/sec^2}{\sin 12° \cos 12°}}$

$\approx \boxed{125.44\,ft/sec}$