12.4: Tangent Vectors and Normal Vectors

The unit tangent vector:

Definition: The Unit Tangent Vector

Let C be a smooth curve represented by **r** on an open interval I. The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \text{ provided } \mathbf{r}'(t) \neq \mathbf{0}.$$

Note: If r represents the position of a particle, then $\mathbf{r}'(t)$ is the velocity vector.

Example 1: Find the unit tangent vector for $\mathbf{r}(t) = (2\sin t)\,\mathbf{i} + (2\cos t)\,\mathbf{j} + (4\sin^2 5)\,\mathbf{k}$ at the point $P(1, \sqrt{3}, 1)$. Find a set of parametric equations for the line tangent to the space curve at point *P*.

The principal unit normal vector:

There are infinitely many vectors orthogonal to the unit tangent vector $\mathbf{T}(t)$. One of them is $\mathbf{T}'(t)$.

Why is $\mathbf{T}(t) \perp \mathbf{T}'(t)$?

If we normalize $\mathbf{T}'(t)$, we get the *principal unit normal vector*.

Definition: Principal Unit Normal Vector

Let C be a smooth curve represented by r on an open interval. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the principal unit normal vector at t is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\left\|\mathbf{T}'(t)\right\|}.$$

Example 2: Calculate the unit tangent vector and the principal unit normal vector for $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 3t \rangle$.

<u>Example 3:</u> Calculate the unit tangent vector and the principal unit normal vector for $\mathbf{r}(t) = \left\langle t, t^2 \right\rangle$.

Tangential and normal components of acceleration:

Recall: If $\mathbf{r}(t)\cdot\mathbf{r}(t)=\|\mathbf{r}(t)\|^2=c$, then $\mathbf{r}(t)\cdot\mathbf{r}'(t)=\mathbf{0}$.

Thus, if the velocity is constant, then the velocity and acceleration vectors must be orthogonal. In other words, if the speed $\|\mathbf{r}'(t)\|$ is constant, then $\mathbf{r}'(t)\cdot\mathbf{r}''(t)=\mathbf{0}$.

If an object is not traveling at a constant speed, the velocity and acceleration vectors are not necessarily orthogonal.

The acceleration vector can be broken down into the components: a tangential component acting in the direction of the line of motion, and a normal component acting perpendicular to the line of motion.

Theorem: If $\mathbf{r}(t)$ is the position vector for a smooth curve, and if the principal unit normal vector $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t)$ lies in the same plane as $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

This theorem follows from the fact that $\mathbf{a}(t)$ can be written as a linear combination of $\mathbf{T}(t)$ and $\mathbf{N}(t)$. In other words,

$$\mathbf{a}(t)=a_{\mathbf{T}}\mathbf{T}(t)+a_{\mathbf{N}}\mathbf{N}(t).$$

$a_{\mathbf{T}}$ is the tangential component of acceleration; $a_{\mathbf{N}}$ is the normal component of acceleration.

Theorem: If $\mathbf{r}(t)$ is the position vector for a smooth curve, and if $\mathbf{N}(t)$ exists, then the tangential and normal components of acceleration are as follows:

$$a_{\mathbf{T}}=\frac{d}{dt}\|\mathbf{v}\|=\mathbf{a}\cdot\mathbf{T}=\frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$$

$$a_{\mathbf{N}}=\|\mathbf{v}\|\|\mathbf{T}'\|=\mathbf{a}\cdot\mathbf{N}=\frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|}=\sqrt{\|\mathbf{a}\|^2-a_{\mathbf{T}}^2}$$

The easiest method for finding $\mathbf{N}(t)$ is usually to calculate the other four values in $\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$ and solve for $\mathbf{N}(t)$ algebraically.

Example 4: Suppose that $\mathbf{r}(t) = \left\langle t^3, 2t, 4t^2 \right\rangle$. Calculate $\mathbf{a}(t)$, $a_{\mathbf{T}}$, $a_{\mathbf{N}}$, $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for $t = 1$.