12.4: Tangent Vectors and Normal Vectors

The unit tangent vector:

> Definition: The Unit Tangent Vector
>
> Let C be a smooth curve represented by **r** on an open interval I. The unit tangent vector is
>
> $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \text{ provided } \mathbf{r}'(t) \neq \mathbf{0}.$$

Note: If r represents the position of a particle, then $\mathbf{r}'(t)$ is the velocity vector.

Example 1: Find the unit tangent vector for $\mathbf{r}(t) = (2\sin t)\,\mathbf{i} + (2\cos t)\,\mathbf{j} + (\cancel{4\sin^2 5})\,\mathbf{k}$ $4\sin^2 t\,\hat{k}$ at the point $P(1, \sqrt{3}, 1)$. Find a set of parametric equations for the line tangent to the space curve at point *P*.

$\vec{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t \rangle$

$\vec{r}'(t) = \langle 2\cos t, -2\sin t, 8\sin t\cos t \rangle$

$\|\vec{r}'(t)\| = \sqrt{4\cos^2 t + 4\sin^2 t + 64\sin^2 t\cos^2 t}$

$= \sqrt{4(\cos^2 t + \sin^2 t) + 64\sin^2 t\cos^2 t} = \sqrt{4 + 64\sin^2 t\cos^2 t}$

What t produces $P(1, \sqrt{3}, 1)$? x-coordinate: $2\sin t = 1$, $\sin t = 1/2 \Rightarrow t = \pi/6$

y-coordinate $= 2\cos \pi/6 = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$ ✓

z-coordinate $= 4\sin^2(\pi/6) = 4\left(\frac{1}{2}\right)^2 = 1$ ✓

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The principal unit normal vector:

There are infinitely many vectors orthogonal to the unit tangent vector $\mathbf{T}(t)$. One of them is $\mathbf{T}'(t)$.

Why is $\mathbf{T}(t) \perp \mathbf{T}'(t)$?

$\vec{T}(t) \cdot \vec{T}(t) = \|\vec{T}(t)\|^2 = 1$ (it's a unit vector)

Property #7 of derivative: If $\vec{r}(t) \cdot \vec{r}(t) = c$, then $\vec{r}(t) \cdot \vec{r}'(t) = 0$

Because $\vec{T}(t) \cdot \vec{T}(t)$ is constant, $\vec{T}(t) \cdot \vec{T}'(t) = 0$

and so $\vec{T}(t) \perp \vec{T}'(t)$

Ex 1 cont'd: Put $t = \frac{\pi}{6}$ into $\vec{r}'(t)$ and $\|\vec{r}'\|$.

$$\vec{r}'(t) = \langle 2\cos t, -2\sin t, 8\sin t\cos t \rangle$$

$$\vec{r}'\left(\frac{\pi}{6}\right) = \left\langle 2\cos\frac{\pi}{6}, -2\sin\frac{\pi}{6}, 8\sin\frac{\pi}{6}\cos\frac{\pi}{6} \right\rangle$$

$$= \left\langle 2\frac{\sqrt{3}}{2}, -2\left(\frac{1}{2}\right), 8\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \right\rangle$$

$$= \langle \sqrt{3}, -1, 2\sqrt{3} \rangle$$

$$\left\|\vec{r}'\left(\frac{\pi}{6}\right)\right\| = \sqrt{3+1+4\cdot 3} = \sqrt{16} = 4$$

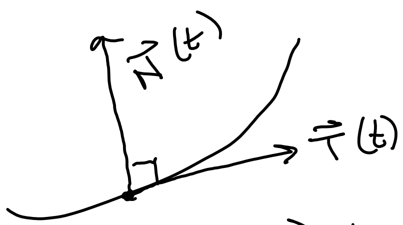
$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle \sqrt{3}, -1, 2\sqrt{3} \rangle}{4} = \boxed{\left\langle \frac{\sqrt{3}}{4}, -\frac{1}{4}, \frac{\sqrt{3}}{2} \right\rangle}$$

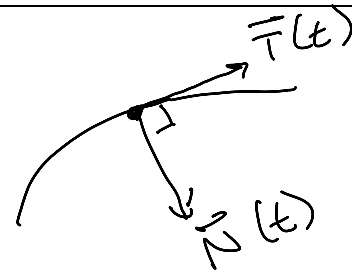
If we normalize $\mathbf{T}'(t)$, we get the *principal unit normal vector*.

<u>Definition</u>: Principal Unit Normal Vector

Let C be a smooth curve represented by r on an open interval. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the principal unit normal vector at t is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$





$\vec{N}(t) \cdot \vec{T}(t) = 0$

Example 2: Calculate the unit tangent vector and the principal unit normal vector for $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 3t \rangle$. (in terms of t)

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 3 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 9} = \sqrt{4(\underbrace{\sin^2 t + \cos^2 t}_{1}) + 9}$

$= \sqrt{4(1) + 9} = \sqrt{13}$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{\langle -2\sin t, 2\cos t, 3 \rangle}{\sqrt{13}} = \boxed{\left\langle -\frac{2}{\sqrt{13}}\sin t, \frac{2}{\sqrt{13}}\cos t, \frac{3}{\sqrt{13}} \right\rangle = \vec{T}(t) \text{ (unit tangent vector)}}$

$\vec{T}\,'(t) = \left\langle -\frac{2}{\sqrt{13}}\cos t, -\frac{2}{\sqrt{13}}\sin t, 0 \right\rangle$

$\|\vec{T}\,'(t)\| = \sqrt{\frac{4}{13}\cos^2 t + \frac{4}{13}\sin^2 t + 0^2} = \sqrt{\frac{4}{13}(\underbrace{\cos^2 t + \sin^2 t}_{1})}$

$= \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}}$

$\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \dfrac{\left\langle -\frac{2}{\sqrt{13}}\cos t, -\frac{2}{\sqrt{13}}\sin t, 0 \right\rangle}{\frac{2}{\sqrt{13}}} = \boxed{\langle -\cos t, -\sin t, 0 \rangle = \vec{N}(t) \text{ (principal unit normal vector)}}$

Example 3: Calculate the unit tangent vector and the principal unit normal vector for $\mathbf{r}(t) = \langle t, t^2 \rangle$.

$\vec{r}(t) = \langle t, t^2 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1^2 + (2t)^2} = \sqrt{1+4t^2}$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle 1, 2t \rangle}{\sqrt{1+4t^2}} = \boxed{\left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle = \vec{T}(t) \text{ (unit tangent vector)}}$$

Write $\vec{T}(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle$

$= (1+4t^2)^{-1/2} \langle 1, 2t \rangle$

Use product rule (Derivative property #3)

$$\vec{T}\,'(t) = (1+4t^2)^{-1/2} \langle 0, 2 \rangle + \left(-\tfrac{1}{2}\right)(1+4t^2)^{-3/2}(8t)\langle 1, 2t \rangle$$

$$= \left\langle 0, \frac{2}{\sqrt{1+4t^2}} \right\rangle - \frac{4t}{(1+4t^2)^{3/2}} \langle 1, 2t \rangle$$

$$= \left\langle 0, \frac{2}{\sqrt{1+4t^2}} \right\rangle + \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, -\frac{8t^2}{(1+4t^2)^{3/2}} \right\rangle$$

$$= \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{\sqrt{1+4t^2}}\left(\frac{1+4t^2}{1+4t^2}\right) - \frac{8t^2}{(1+4t^2)^{3/2}} \right\rangle$$

$$= \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2+8t^2-8t^2}{(1+4t^2)^{3/2}} \right\rangle$$

$$= \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle$$

$$\|\vec{T}\,'(t)\| = \sqrt{\frac{16t^2}{(1+4t^2)^3} + \frac{4}{(1+4t^2)^3}} = \sqrt{\frac{16t^2+4}{(1+4t^2)^3}}$$

$$= \sqrt{\frac{4(4t^2+1)}{(1+4t^2)^3}} = \sqrt{\frac{4}{(1+4t^2)^2}} = \frac{2}{1+4t^2}$$

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{\left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle}{\frac{2}{1+4t^2}}$$

$$= \frac{1+4t^2}{2} \left\langle \frac{-4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle = \boxed{\left\langle \frac{-2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}} \right\rangle}$$

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Notice:

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle$$

$$\vec{N}(t) = \left\langle -\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}} \right\rangle$$

For plane curves, if $\vec{T}(t) = \langle x(t), y(t) \rangle$,

then $\vec{N}(t)$ must be either

$\vec{N}_1(t) = \langle -y(t), x(t) \rangle$ or $N_2(t) = \langle y(t), -x(t) \rangle$.

(Because $\vec{N}(t) \cdot \vec{T}(t) = 0$ and their magnitudes are both 1.

One of $\vec{N}_1$ or $\vec{N}_2$ will point toward the "inside" (concave side) of the curve. The other will point outward.

For $\vec{N}$, we want the one pointing inward.

Tangential and normal components of acceleration:

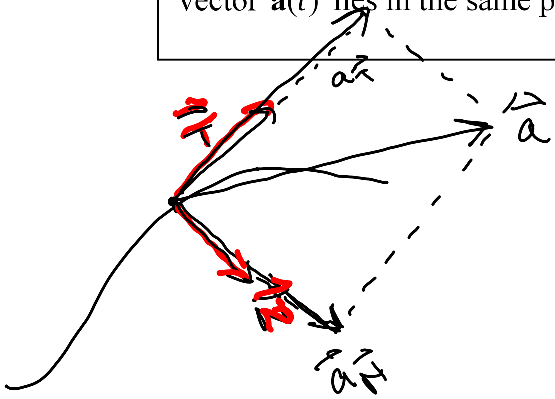
<u>Recall</u>: If $\mathbf{r}(t)\cdot\mathbf{r}(t)=\|\mathbf{r}(t)\|^2=c$, then $\mathbf{r}(t)\cdot\mathbf{r}'(t)=\mathbf{0}$.

Thus, if the velocity is constant, then the velocity and acceleration vectors must be orthogonal. In other words, if the speed $\|\mathbf{r}'(t)\|$ is constant, then $\mathbf{r}'(t)\cdot\mathbf{r}''(t)=\mathbf{0}$.

If an object is not traveling at a constant speed, the velocity and acceleration vectors are not necessarily orthogonal.

The acceleration vector can be broken down into the components: a tangential component acting in the direction of the line of motion, and a normal component acting perpendicular to the line of motion.

> <u>Theorem</u>: If $\mathbf{r}(t)$ is the position vector for a smooth curve, and if the principal unit normal vector $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t)$ lies in the same plane as $\mathbf{T}(t)$ and $\mathbf{N}(t)$.



$$\vec a(t) = a_{\vec T}\,\vec T(t) + a_{\vec N}\,\vec N(t)$$

$$\left.\begin{array}{l} a_{\vec T} = \text{tangential component of acceleration} \\ a_{\vec N} = \text{normal component of acceleration} \end{array}\right\}\ \text{scalars}$$

This theorem follows from the fact that $\mathbf{a}(t)$ can be written as a linear combination of $\mathbf{T}(t)$ and $\mathbf{N}(t)$. In other words,

$$\mathbf{a}(t)=a_{\mathrm{T}}\mathbf{T}(t)+a_{\mathrm{N}}\mathbf{N}(t).$$

a_{T} is the tangential component of acceleration; a_{N} is the normal component of acceleration.

> <u>Theorem</u>: If $\mathbf{r}(t)$ is the position vector for a smooth curve, and if $\mathbf{N}(t)$ exists, then the tangential and normal components of acceleration are as follows:
>
> $$a_{\mathrm{T}}=\frac{d}{dt}\|\mathbf{v}\|=\mathbf{a}\cdot\mathbf{T}=\frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$$
>
> $$a_{\mathrm{N}}=\|\mathbf{v}\|\|\mathbf{T}'\|=\mathbf{a}\cdot\mathbf{N}=\frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|}=\sqrt{\|\mathbf{a}\|^2-a_{\mathrm{T}}^2}$$

The easiest method for finding $\mathbf{N}(t)$ is usually to calculate the other four values in $\mathbf{a}(t) = a_{\mathrm{T}}\mathbf{T}(t) + a_{\mathrm{N}}\mathbf{N}(t)$ and solve for $\mathbf{N}(t)$ algebraically.

Example 4: Suppose that $\mathbf{r}(t) = \langle t^3, 2t, 4t^2 \rangle$. Calculate $\mathbf{a}(t)$, a_{T}, a_{N}, $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for $t = 1$.

$\vec{r}\,'(t) = \langle 3t^2, 2, 8t \rangle = \vec{v}(t)$

$\vec{r}\,'(1) = \langle 3(1)^2, 2, 8(1) \rangle = \langle 3, 2, 8 \rangle = \vec{v}(1)$

$\|\vec{r}\,'(1)\| = \sqrt{3^2+2^2+8^2} = \sqrt{9+4+64} = \sqrt{77}$

$\vec{T}(1) = \dfrac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|} = \dfrac{\langle 3,2,8\rangle}{\sqrt{77}} = \boxed{\left\langle \dfrac{3}{\sqrt{77}}, \dfrac{2}{\sqrt{77}}, \dfrac{8}{\sqrt{77}} \right\rangle = \vec{T}(1)}$

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \dfrac{d}{dt}\langle 3t^2, 2, 8t \rangle$

$= \langle 6t, 0, 8 \rangle$

$\vec{a}(1) = \langle 6(1), 0, 8 \rangle = \boxed{\langle 6, 0, 8 \rangle = \vec{a}(1)}$

$a_T(1) = \vec{a}(1) \cdot \vec{T}(1)$

$= \langle 6, 0, 8 \rangle \cdot \left\langle \dfrac{3}{\sqrt{77}}, \dfrac{2}{\sqrt{77}}, \dfrac{8}{\sqrt{77}} \right\rangle$

$= \dfrac{18}{\sqrt{77}} + 0 + \dfrac{64}{\sqrt{77}} = \boxed{\dfrac{82}{\sqrt{77}} = a_T(1)}$

$a_N = \dfrac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}$

$\vec{v}(1) \times \vec{a}(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 8 \\ 6 & 0 & 8 \end{vmatrix} = \hat{i}(16-0) - \hat{j}(24-48) + \hat{k}(0-12)$

$= \langle 16, 24, -12 \rangle$

$a_N(1) = \dfrac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|} = \dfrac{\|\langle 16, 24, -12\rangle\|}{\|\langle 3,2,8\rangle\|} = \boxed{\dfrac{\sqrt{976}}{\sqrt{77}} = a_N}$

$\vec{a}(1) = a_T(1)\vec{T}(1) + a_N \vec{N}(1)$

$\langle 6, 0, 8 \rangle = \dfrac{82}{\sqrt{77}} \left\langle \dfrac{3}{\sqrt{77}}, \dfrac{2}{\sqrt{77}}, \dfrac{8}{\sqrt{77}} \right\rangle + \dfrac{\sqrt{976}}{\sqrt{77}} \vec{N}(1)$

$\langle 6, 0, 8 \rangle = \left\langle \dfrac{246}{77}, \dfrac{164}{77}, \dfrac{656}{77} \right\rangle + \dfrac{\sqrt{976}}{\sqrt{77}} \vec{N}(1)$

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$$\left\langle \frac{216}{77}, -\frac{164}{77}, -\frac{40}{77} \right\rangle = \frac{\sqrt{976}}{\sqrt{77}} \vec{N}(1)$$

$$\boxed{\frac{\sqrt{77}}{\sqrt{976}} \left\langle \frac{216}{77}, \frac{-164}{77}, \frac{-40}{77} \right\rangle} = \vec{N}(1)$$