12.5: Arc Length and Curvature

Arc length:

Theorem: Arc length of a curve in $\mathbb{R}^3$

Suppose C is a smooth curve described by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ on an interval $[a,b]$, and that the curve is traversed exactly once as t increases from a to b.

The arc length of C on the interval is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|\,dt .$$

Example 1: Find the arc length of the curve described by $\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$ over the interval $[0,2]$.

$\vec{r}(t) = \langle 1, t^2, t^3 \rangle$

$\vec{r}\,'(t) = \langle 0, 2t, 3t^2 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{0^2 + (2t)^2 + (3t^2)^2} = \sqrt{4t^2 + 9t^4}$

Note: $\sqrt{t^2} = |t|$ generally. Here, $t \geq 0$, so $\sqrt{t^2} = t$

$s = \int_0^2 \sqrt{4t^2 + 9t^4}\,dt = \int_0^2 \sqrt{t^2(4 + 9t^2)}\,dt$

$= \int_0^2 \sqrt{t^2}\sqrt{4 + 9t^2}\,dt = \int_0^2 t\,(4 + 9t^2)^{1/2}\,dt = \frac{1}{18}\int_4^{40} u^{1/2}\,du$

$u = 9t^2 + 4$

$\frac{1}{18}\,du = dt$

$t = 0 \Rightarrow u = 4$

$t = 2 \Rightarrow u = 40$

$= \frac{1}{18} \cdot \frac{u^{3/2}}{3/2}\Big|_4^{40} = \frac{2}{18 \cdot 3}\left[40^{3/2} - 4^{3/2}\right] \approx \boxed{9.0736}$

$\frac{1}{27}\left[40^{3/2} - 4^{3/2}\right]$

Example 2: Find the arc length of the curve described by $\mathbf{r}(t) = \langle 6t, 4\sin t, 4\cos t \rangle$ over the interval $[0, 2\pi]$.

$\vec{r}\,'(t) = \langle 6, 4\cos t, -4\sin t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{36 + 16\cos^2 t + 16\sin^2 t} = \sqrt{36 + 16(1)} = \sqrt{52}$

$s = \int_0^{2\pi} \sqrt{52}\,dt = \sqrt{52}\,t\Big|_0^{2\pi} = \sqrt{52}\,(2\pi - 0)$

$= 2\pi\sqrt{52} = 2\pi(2\sqrt{13}) = \boxed{4\pi\sqrt{13}}$

≈ 45.3

Definition:

Suppose C is a smooth curve described by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ on an interval $[t_0, b]$. Then, for $t \in [t_0, b]$, the *arc length parameter with base point* $P(t_0)$, or *arc length function*, is

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du = \int_{t_0}^{t} \|\mathbf{r}'(u)\|\, du .$$

Note: The arc length parameter function $s(t)$ is nonnegative and increases with increasing t. It tells us the distance along the curve from a fixed base point $P(t_0)$ to the point $P(t)$. Because t is being used as the input for the function $s(t)$, we must use another variable (e.g., u) as the variable of integration.

Example 3: Find the arc length parameter for the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, using $t_0 = 0$ as the base point. Write the vector-valued function for the helix in terms of the arc length parameter s. What is $\|r'(s)\|$?

Change variables: $\vec{r}(u) = \langle \cos u \ \sin u, u \rangle$

$\vec{r}\,'(u) = \langle -\sin u, \cos u, 1 \rangle$

$s(t) = \int_0^t \|\vec{r}\,'(u)\|\, du = \int_0^t \sqrt{\underbrace{\sin^2 u + \cos^2 u}_{1} + 1^2}\, du = \int_0^t \sqrt{2}\, du$

$= \sqrt{2}\, u \Big|_0^t = \sqrt{2}(t - 0) = \sqrt{2}t$

$\boxed{s(t) = \sqrt{2}(t) \text{ arc length parameter (arc length function)}}$

Reparametrize $\vec{r}$ using s instead of t: $s = \sqrt{2}\, t \Rightarrow t = \frac{s}{\sqrt{2}}$

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle \Rightarrow \boxed{\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle}$

Find $\|\vec{r}\,'(s)\|$:

$\|\vec{r}\,'(s)\| = \left\| \left\langle -\frac{1}{\sqrt{2}} \sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \right\rangle \right\|$

$= \sqrt{\frac{1}{2}\sin^2\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{2}\cos^2\left(\frac{s}{\sqrt{2}}\right) + \frac{1}{2}} = \sqrt{\frac{1}{2}\Big(\underbrace{\sin^2\left(\frac{s}{\sqrt{2}}\right) + \cos^2\left(\frac{s}{\sqrt{2}}\right)}_{1}\Big) + \frac{1}{2}}$

$= \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = \boxed{1}$ $\quad \vec{r}\,'(s)$ is a unit vector

When s is the arc length parameter, $\|\vec{r}\,'(s)\| = 1$ always

For any parametrization $\vec{r}(s)$ that results in $\|\vec{r}\,'(s)\| = 1$, then s must be the arc length parameter.

Curvature:

Curvature is a measure of how sharply a curve bends. How much does the unit tangent vector change for each unit of arc length? Remember, the unit tangent vector has magnitude 1.

> Definition: Let C be a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, described by $\mathbf{r}(s)$ where s is the arc length parameter and $\mathbf{T}(s)$ is the unit tangent vector with respect to the arch length s. Then the curvature κ at s is
>
> $$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \mathbf{T}'(s) \right\|.$$
>
> Kappa

Example 4: Find the curvature of a circle with radius r.

$\vec{r}(t) = \langle r\cos t, r\sin t\rangle$ Need the arc length parameter s.

Change variables: $\vec{r}(u) = \langle r\cos u, r\sin u\rangle$

$\vec{r}\,'(u) = \langle -r\sin u, r\cos u\rangle$

$\|\vec{r}\,'(u)\| = \sqrt{r^2\sin^2 u + r^2\cos^2 u} = \sqrt{r^2} = r$

Arc length function (arc length parameter): $s(t) = \int_0^t \|\vec{r}\,'(u)\|\,du = \int_0^t r\,du = ru\Big|_0^t = rt - r(0) = rt$

Recall from Trig: Arc length Formula: $s = r\theta$

$s = rt \Rightarrow t = \frac{s}{r}$. Reparametrize using s

$\vec{r}(t) = \langle r\cos t, r\sin t\rangle \Rightarrow \vec{r}(s) = \langle r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right)\rangle$

$\vec{r}\,'(s) = \langle r\left(-\frac{1}{r}\sin\left(\frac{s}{r}\right)\right), r\left(\frac{1}{r}\cos\left(\frac{s}{r}\right)\right)\rangle = \langle -\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\rangle$

For many functions, it is not possible or not practical to rewrite the position function in terms of the arc length parameter. Fortunately, there are other ways to calculate the curvature (keeping the function in terms of the original parameter t).

$\vec{T}(s) = \frac{\vec{r}\,'(s)}{\|\vec{r}\,'(s)\|} = \frac{\langle -\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\rangle}{\sqrt{\sin^2\left(\frac{s}{r}\right) + \cos^2\left(\frac{s}{r}\right)}} = \frac{\langle -\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\rangle}{1} = \langle -\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\rangle$

> Theorem: If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature κ of C is
>
> $$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

$\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \|T'(s)\|$

$= \left\| \langle -\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right)\rangle \right\| = \sqrt{\frac{1}{r^2}\cos^2\left(\frac{s}{r}\right) + \frac{1}{r^2}\sin^2\left(\frac{s}{r}\right)}$

$= \sqrt{\frac{1}{r^2}} = \boxed{\frac{1}{r}}$

Important: Curvature of a circle of radius r is $\boxed{K = \frac{1}{r}}$

Example 5: Find the curvature of the curve given by $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$ at the point where $t = 1$.

$\vec{r}(t) = \langle t, t^2, 0 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t, 0 \rangle$

$\vec{r}\,''(t) = \langle 0, 2, 0 \rangle$

we'll use $K = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$

$\vec{r}\,'(1) = \langle 1, 2, 0 \rangle$

$\vec{r}\,''(1) = \langle 0, 2, 0 \rangle$

$$\vec{r}\,'(1) \times \vec{r}\,''(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 2 \rangle$$

$$K(1) = \frac{\|\vec{r}\,'(1) \times \vec{r}\,''(1)\|}{\|\vec{r}\,'(1)\|^3} = \frac{\sqrt{0^2+0^2+2^2}}{\left(\sqrt{1^2+2^2+0^2}\right)^3} = \frac{2}{(\sqrt{5})^3} = \frac{2}{\sqrt{5}\sqrt{5}\sqrt{5}} = \boxed{\frac{2}{5\sqrt{5}}}$$

Example 6: Find the curvature of the curve given by $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + t\,\mathbf{k}$ at the point $P(-4, 0, \pi)$.

$\vec{r}(t) = \langle 4\cos t, 3\sin t, t \rangle$

At $P(-4, 0, \pi)$, $t = \pi$

$\vec{r}\,'(t) = \langle -4\sin t, 3\cos t, 1 \rangle$

$\vec{r}\,''(t) = \langle -4\cos t, -3\sin t, 0 \rangle$

$\vec{r}\,'(\pi) = \langle -4\sin\pi, 3\cos\pi, 1 \rangle = \langle 0, -3, 1 \rangle$

$\vec{r}\,''(\pi) = \langle -4\cos\pi, -3\sin\pi, 0 \rangle = \langle 4, 0, 0 \rangle$

$$\vec{r}\,'(\pi) \times \vec{r}\,''(\pi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -3 & 1 \\ 4 & 0 & 0 \end{vmatrix} = \langle 0, 4, 12 \rangle$$

$$\|\vec{r}\,'(\pi) \times \vec{r}\,''(\pi)\| = \sqrt{4^2+12^2} = \sqrt{16+144} = \sqrt{160} = \sqrt{16 \cdot 10} = 4\sqrt{10}$$

$$K = \frac{\|\vec{r}\,'(\pi) \times \vec{r}\,''(\pi)\|}{\|\vec{r}\,'(\pi)\|^3} = \frac{4\sqrt{10}}{\left(\sqrt{9+1}\right)^3} = \frac{4\sqrt{10}}{(\sqrt{10})^3} = \frac{4}{(\sqrt{10})^2} = \frac{4}{10} = \boxed{\frac{2}{5}}$$

Example 7: Find the curvature of the curve given by $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, 2 \rangle$.

$\vec{r}(t) = \langle e^t \cos t, e^t \sin t, 2 \rangle$

$\vec{r}\,'(t) = \langle -e^t \sin t + e^t \cos t, e^t \cos t + e^t \sin t, 0 \rangle$

$\vec{r}\,''(t) = \langle -\cancel{e^t \cos t} - e^t \sin t - e^t \sin t + \cancel{e^t \cos t}, -\cancel{e^t \sin t} + e^t \cos t + e^t \cos t + \cancel{e^t \sin t}, 0 \rangle$

$= \langle -2e^t \sin t, 2e^t \cos t, 0 \rangle$

$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \left\| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ -e^t \sin t + e^t \cos t & e^t \cos t + e^t \sin t & 0 \\ -2e^t \sin t & 2e^t \cos t & 0 \end{array} \right\|$$

$= \| \langle 0, 0, -\cancel{2e^{2t} \sin t \cos t} + 2e^{2t} \cos^2 t + \cancel{2e^{2t} \sin t \cos t} + 2e^{2t} \sin^2 t \rangle \|$

$= \| \langle 0, 0, 2e^{2t} \cos^2 t + 2e^{2t} \sin^2 t \rangle \|$

$= \| \langle 0, 0, 2e^{2t} (1) \rangle \| = \| \langle 0, 0, 2e^{2t} \rangle \| = 2e^{2t}$

$\|\vec{r}\,'(t)\| = \| \langle -e^t \sin t + e^t \cos t, e^t \cos t + e^t \sin t, 0 \rangle \|$

$= \sqrt{(-e^t \sin t + e^t \cos t)^2 + (e^t \cos t + e^t \sin t)^2 + 0^2}$

See next page

Curvature in rectangular coordinates:

If C is the graph of a twice-differentiable function $y = f(x)$, then the curvature at the point (x, y) is

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

Example 8: Find the curvature of the graph of $f(x) = 2x + \frac{4}{x}$ at the point where $x = 1$.

$$\left.\begin{array}{l} f(x) = 2x + 4x^{-1} \\ f'(x) = 2 - 4x^{-2} \\ f''(x) = 8x^{-3} \end{array}\right\} \Rightarrow \begin{array}{l} f'(1) = 2 - \frac{4}{(1)^2} = 2 - 4 = -2 \\ f''(1) = \frac{8}{(1)^3} = 8 \end{array}$$

$$K = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} \Rightarrow K(1) = \frac{|8|}{[1 + (-2)^2]^{3/2}} = \frac{8}{5^{3/2}} = \frac{8}{5 \cdot 5^{1/2}}$$

$$= \boxed{\frac{8}{5\sqrt{5}}}$$

Ex 7 cont'd:

$$\|\vec{r}\,'(t)\| = \| \langle -e^t \sin t + e^t \cos t,\ e^t \cos t + e^t \sin t,\ 0 \rangle \|$$

$$= \sqrt{(-e^t \sin t + e^t \cos t)^2 + (e^t \cos t + e^t \sin t)^2 + 0^2}$$

$$= \sqrt{e^{2t}\sin^2 t - \cancel{2e^{2t}\sin t \cos t} + e^{2t}\cos^2 t + e^{2t}\cos^2 t + \cancel{2e^{2t}\cos t \sin t} + e^{2t}\sin^2 t}$$

$$= \sqrt{2e^{2t}\sin^2 t + 2e^{2t}\cos^2 t}$$

$$= \sqrt{2e^{2t}(\sin^2 t + \cos^2 t)} = \sqrt{2e^{2t}(1)} = \sqrt{2}\sqrt{e^{2t}}$$

$$= \sqrt{2}\, e^t$$

$$K = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{2e^{2t}}{(\sqrt{2}\, e^t)^3} = \frac{2e^{2t}}{2^{3/2} e^{3t}}$$

$$= \frac{1}{2^{1/2} e^t} = \boxed{\frac{1}{\sqrt{2}\, e^t}} = \boxed{\frac{\sqrt{2}}{2e^t}}$$

Example 9: Find the point on the curve at which the curvature is at a maximum.

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$$\kappa(x) = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}} = \frac{|e^x|}{\left[1+(e^x)^2\right]^{3/2}} = \frac{e^x}{(1+e^{2x})^{3/2}}$$

To find where $\kappa(x)$ is maximized, find its derivative.

$$\kappa'(x) = \frac{(1+e^{2x})^{3/2}e^x - e^x\left(\frac{3}{2}\right)(1+e^{2x})^{1/2}(2e^{2x})}{\left[(1+e^{2x})^{3/2}\right]^2} = \frac{e^x(1+e^{2x})^{3/2} - 3e^{3x}(1+e^{2x})^{1/2}}{(1+e^{2x})^3}$$

$$= \frac{(1+e^{2x})^{1/2}\left[e^x(1+e^{2x}) - 3e^{3x}\right]}{(1+e^{2x})^3}$$

[Factor out $(1+e^{2x})^{1/2}$]

$$= \frac{e^x + e^{3x} - 3e^{3x}}{(1+e^{2x})^{5/2}} = \frac{-2e^{3x}+e^x}{(1+e^{2x})^{3/2}} = \frac{e^x(-2e^{2x}+1)}{(1+e^{2x})^{3/2}}$$

Set numerator $= 0$:

$e^x = 0$ never true, or $-2e^{2x}+1=0$

$1 = 2e^{2x}$

$\frac{1}{2} = e^{2x}$

$\ln\left(\frac{1}{2}\right) = \ln(e^{2x})$

$\ln\left(\frac{1}{2}\right) = 2x$

$\frac{\ln\left(\frac{1}{2}\right)}{2} = x$

max curvature when $\boxed{x = -\frac{1}{2}\ln 2}$

Relationship of curvature to the tangential and normal components of acceleration:

Tangential component of acceleration: rate of change of speed (thus rate of change of arc length).

Normal component of acceleration: involves both rate of change of speed, and also curvature.

> Theorem: If $\mathbf{r}(t)$ is the position vector for a smooth curve C, then the acceleration vector is
>
> $$a(t) = a_{\mathrm{T}}\mathbf{T}(t) + a_{\mathrm{N}}\mathbf{N}(t)$$
> $$= \frac{d^2s}{dt^2}\mathbf{T}(t) + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}(t).$$

$\frac{ds}{dt}$ should be squared!

Example 10: Find the tangential and normal components of the acceleration for the curve given by $\mathbf{r}(t) = \langle e^t\cos t, e^t\sin t, 2\rangle$. (Same curve as Example 7.)

$\frac{ds}{dt} = \|\vec{r}\,'(t)\| = e^t\sqrt{2}$ (from Ex 7)

(speed)

$\frac{d^2s}{dt^2} = e^t\sqrt{2} \Rightarrow a_{\vec{T}} = \frac{d^2s}{dt^2} = \boxed{e^t\sqrt{2}}$

$a_{\vec{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \frac{1}{e^t\sqrt{2}}\left(e^t\sqrt{2}\right)^2 = \boxed{e^t\sqrt{2}}$

Ex 7 $\Rightarrow \kappa = \frac{1}{e^t\sqrt{2}}$

More on motion in $\mathbb{R}^3$:

Motion in $\mathbb{R}^3$ can be described using the *Frenet-Serret frame* or *TNB frame*. The TNB is composed of the unit tangent vector **T**, the principal unit normal vector **B**, and the binormal vector, which is the unit vector orthogonal to both **B** and **T**. These three mutually orthogonal unit vectors form a *basis* of $\mathbb{R}^3$, meaning any other vector can be written as a linear combination of these three vectors.

Useful site: https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

Normal plane to curve *C* at the point *P*: the plane determined by **N** and **B**. The vector **T** always emerges from the normal plane at a right angle.

Osculating plane to curve *C* at the point *P*: The plane determined by **T** and **N**. The osculating plane is the plane that comes closest to containing the part of the curve near *P*.

Osculating circle to curve *C* at the point *P*: Lies in the osculating plane. This circle shares the same tangent vector, normal vector, and curvature as the curve *C* at *P*. (Thus, for the part of the curve near *P*, the osculating circle closely approximates the curve.)

Torsion τ : measure of a curve's tendency to not be planar.

Curvature κ: measure of a curve's tendency to not be a straight line.

More formulas (you aren't responsible for these, but may use them if you like. Sometimes they are easier.)

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\left\|\mathbf{r}'(t) \times \mathbf{r}''(t)\right\|}$$

$$\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\left\|\mathbf{r}'(t) \times \mathbf{r}''(t)\right\|} \times \frac{\mathbf{r}'(t)}{\left\|\mathbf{r}'(t)\right\|}$$

$$\tau = -\mathbf{B}'(s) \cdot \mathbf{N}(t) = \frac{\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix}}{\left\|\mathbf{r}'(t) \times \mathbf{r}''(t)\right\|^2}$$