13.10: Lagrange Multipliers

Suppose we want to maximize (or minimize) the function $f(x,y)$ while making sure that $g(x,y)=k$ is true. In this situation, the condition $g(x,y)=k$ is called a *constraint*.

Think of a sequence of level curves $f(x,y)=c_1$, $f(x,y)=c_2$, $f(x,y)=c_3$, etc. At the optimal value of c_n, the curve $f(x,y)=c_n$ will "barely touch" the curve $g(x,y)=k$. We want to find the points where $f(x,y)=c_n$ and $g(x,y)=k$ share a common tangent line. For the curves to share a common tangent line, their normal lines at the point of tangency $P(x_0,y_0)$ must be identical. Therefore, their gradients must be parallel.

We can write $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$, where λ is a constant called a Lagrange multiplier.

If we extend the concept to functions on $\mathbb{R}^3$, we optimize $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$. In this case, the optimal point is point of common tangency between some level surface $f(x,y,z)=c_n$ of $f(x,y,z)$, and the surface $g(x,y,z)=k$, which can be thought of as one of the level surfaces of $g(x,y,z)$. At the point of tangency, the gradient vectors $\nabla f(x_0,y_0,z_0)$ and $\nabla g(x_0,y_0,z_0)$ will be parallel.

Lagrange's Theorem: Suppose the functions *f* and *g* have continuous first partial derivatives, and that *f* has an extremum at a point (x_0,y_0) on the smooth constraint curve $g(x,y)=k$.

If $\nabla g(x_0,y_0)\neq\mathbf{0}$, then there exists a real number λ such that $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$.

Note: This can be extended to functions of three variables, in which $f(x,y,z)$ has an extremum on the smooth level surface $g(x,y,z)=k$. In this case, there exists a real number λ such that $\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)$.

The Method of Lagrange Multipliers:

Suppose f and g satisfy the hypotheses of Lagrange's Theorem, that f has a maximum or minimum value satisfying $g(x, y, z) = k$, and that $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$.

Step 1: Find all values of λ , x, and y such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$.

This means you'll have to solve a system of simultaneous equations:

$$\begin{aligned}
&f_x(x, y, z) = \lambda g_x(x, y, z) \\
&f_y(x, y, z) = \lambda g_y(x, y, z) \\
&f_z(x, y, z) = \lambda g_z(x, y, z) \\
&f_z(x, y, z) = \lambda g_z(x, y, z) \\
&g(x, y, z) = k
\end{aligned}$$

Step 2: Evaluate f at each point obtained in Step 1. The largest of these values is the maximum, and the smallest of these values is the minimum.

Example 1: Find the extreme values of $f(x, y) = 4x + 6y$ on the circle $x^2 + y^2 = 13$.

Example 2: Find the maximum value of $P = xy^2z$ subject to the constraint $x + y + z = 32$.

Example 3: Find positive numbers *x* and *y* that minimize $f(x, y) = 3x + y + 10$ subject to the constraint $x^2 y = 6$.

Example 4: A rectangular box without a lid is to be made from 12 square meters of cardboard. Find the maximum volume of such a box.

Example 5: Find the maximum volume of a rectangular box inscribed in the ellipsoid $x^2 + 3y^2 + 4z^2 = 12$.

Example 6: Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$.

Example 7: Find the extreme values of $f(x, y, z) = x^2 y^2 z^2$ on the sphere $x^2 + y^2 + z^2 = 1$.

Optimization problems with two constraints:

Suppose we want to find the extrema of $f(x,y,z)$ subject to <u>two</u> constraints, $g(x,y,z)=k$ and $h(x,y,z)=c$. Then, the gradient of *f* must be a linear combination of the gradients of *g* and *h*. In other words, if *f* has an extremum at (x_0,y_0,z_0), then

$\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)+\mu\nabla h(x_0,y_0,z_0)$, where λ and μ are scalars.

Lagrange's method results in five equations in five variables:

$$
\begin{aligned}
&f_x=\lambda g_x+\mu h_x\\
&f_y=\lambda g_y+\mu h_y\\
&f_z=\lambda g_z+\mu h_z\\
&g(x,y,z)=k\\
&h(x,y,z)=c
\end{aligned}
$$

<u>Example 8:</u> Find the extreme values of $f(x,y,z)=3x-y-3z$ on the curve of intersection of $x+y-z=0$ and $x^2+2z^2=1$.

Example 9: Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2z = 6$ and $x + y = 12$.

Example 10: Suppose the temperature at each point on the sphere $x^2 + y^2 + z^2 = 50$ is given by the function $T(x, y, z) = 100 + x^2 + y^2$. Find the maximum temperature on the curve formed by the intersection of the sphere and the plane $x - z = 0$.