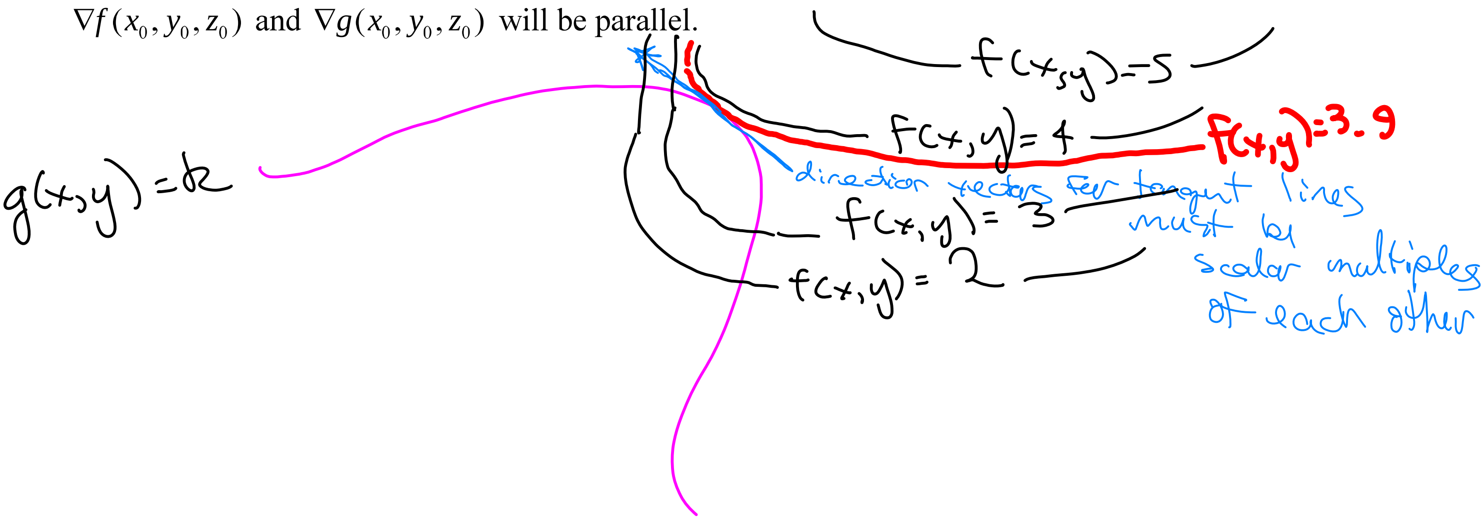
# 13.10: Lagrange Multipliers

Suppose we want to maximize (or minimize) the function $f(x, y)$ while making sure that $g(x, y) = k$ is true. In this situation, the condition $g(x, y) = k$ is called a *constraint*.

Think of a sequence of level curves $f(x, y) = c_1$, $f(x, y) = c_2$, $f(x, y) = c_3$, etc. At the optimal value of $c_n$, the curve $f(x, y) = c_n$ will "barely touch" the curve $g(x, y) = k$. We want to find the points where $f(x, y) = c_n$ and $g(x, y) = k$ share a common tangent line. For the curves to share a common tangent line, their normal lines at the point of tangency $P(x_0, y_0)$ must be identical. Therefore, their gradients must be parallel.

We can write $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$, where $\lambda$ is a constant called a <u>Lagrange multiplier</u>.

If we extend the concept to functions on $\mathbb{R}^3$, we optimize $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. In this case, the optimal point is point of common tangency between some level surface $f(x, y, z) = c_n$ of $f(x, y, z)$, and the surface $g(x, y, z) = k$, which can be thought of as one of the level surfaces of $g(x, y, z)$. At the point of tangency, the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ will be parallel.




<u>Lagrange's Theorem</u>: Suppose the functions *f* and *g* have continuous first partial derivatives, and that *f* has an extremum at a point $(x_0, y_0)$ on the smooth constraint curve $g(x, y) = k$.

lambda

If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there exists a real number $\lambda$ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$.

<u>Note</u>: This can be extended to functions of three variables, in which $f(x, y, z)$ has an extremum on the smooth level surface $g(x, y, z) = k$. In this case, there exists a real number $\lambda$ such that $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$.

The Method of Lagrange Multipliers:

Suppose $f$ and $g$ satisfy the hypotheses of Lagrange's Theorem, that $f$ has a maximum or minimum value satisfying $g(x,y,z)=k$, and that $\nabla g(x_0,y_0,z_0)\neq \mathbf{0}$.

Step 1: Find all values of $\lambda$, $x$, and $y$ $, z$ such that $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ and $g(x,y,z)=k$.

This means you'll have to solve a system of simultaneous equations:

$$f_x(x,y,z)=\lambda g_x(x,y,z)$$
$$f_y(x,y,z)=\lambda g_y(x,y,z)$$
$$f_z(x,y,z)=\lambda g_z(x,y,z)$$
$$\cancel{f_z(x,y,z)=\lambda g_z(x,y,z)}$$
$$g(x,y,z)=k$$

Step 2: Evaluate $f$ at each point obtained in Step 1. The largest of these values is the maximum, and the smallest of these values is the minimum.

**Example 1:** Find the extreme values of $f(x,y)=4x+6y$ on the circle $x^2+y^2=13$.

Find $\nabla g$ and $\nabla f$.

$g(x,y)=x^2+y^2-13$

$\nabla g(x,y)=\langle g_x, g_y\rangle = \langle 2x, 2y\rangle$

$\nabla f(x,y)=\langle f_x, f_y\rangle=\langle 4, 6\rangle$

$\nabla f=\lambda\nabla g$

$\langle 4,6\rangle=\lambda\langle 2x,2y\rangle$

$\Downarrow$

$$\left.\begin{aligned}4&=2\lambda x\\ 6&=2\lambda y\\ x^2+y^2&=13\end{aligned}\right\}\text{ 3 eqns in 3 variables } (x,y,\lambda)$$

$4=2\lambda x \Rightarrow \frac{4}{2\lambda}=x$, $6=2\lambda y\Rightarrow \frac{6}{2\lambda}=y$ $\Rightarrow$ $x=\frac{2}{\lambda}$, $y=\frac{3}{\lambda}$ Put these into $x^2+y^2=13$:

$\left(\frac{2}{\lambda}\right)^2+\left(\frac{3}{\lambda}\right)^2=13$

$\frac{4}{\lambda^2}+\frac{9}{\lambda^2}=13$

$\frac{13}{\lambda^2}=13$

$13=13\lambda^2$

$1=\lambda^2$

$\lambda=\pm 1$

$\underline{\lambda=1}$ $x=\frac{2}{\lambda}=\frac{2}{1}=2$, $y=\frac{3}{\lambda}=\frac{3}{1}=3$ $(2,3)$

$f(2,3)=4(2)+6(3)=8+18=26$

$\underline{\lambda=-1}$ $x=\frac{2}{\lambda}=\frac{2}{-1}=-2$, $y=\frac{3}{\lambda}=\frac{3}{-1}=-3$ $(-2,-3)$

$f(-2,-3)=4(-2)+6(-3)=-8-18=-26$

(Answer on next page)

Ex 1 cont'd:

$$\boxed{\begin{array}{l}\text{The maximum value of } f \text{ is } f(2,3) = 26.\\ \text{The minimum value of } f \text{ is } f(-2,-3) = -26\end{array}}$$

**Example 2:** Find the maximum value of $P = xy^2z$ subject to the constraint $x + y + z = 32$.

$P(x,y,z) = xy^2z$

$g(x,y,z) = 32$

$\nabla P(x,y,z) = \langle y^2z, 2xyz, xy^2 \rangle$

$\nabla g(x,y,z) = \langle 1,1,1 \rangle$

$\nabla P = \lambda \nabla g$

$\Downarrow$

4 eqns, 4 variables

$$\left\{\begin{array}{l} y^2z = \lambda(1) = \lambda \\ 2xyz = \lambda(1) = \lambda \\ xy^2 = \lambda(1) = \lambda \\ x + y + z = 32 \end{array}\right.$$

$y^2z = 2xyz$ (setting $\lambda$'s equal)

$2xyz = xy^2$

Setting $2xyz$'s equal: $y^2z = xy^2$

$y^2z = 2xyz$

Divide by $yz$: (would need $y, z = 0$ for this)

$\dfrac{y^2z}{yz} = \dfrac{2xyz}{yz} \Rightarrow y = 2x$

$2xyz = xy^2$

Divide by $xy$:

$\dfrac{2xyz}{xy} = \dfrac{xy^2}{xy}$

$2z = y$

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**Example 3:** Find positive numbers $x$ and $y$ that minimize $f(x, y) = 3x + y + 10$ subject to the constraint $x^2y = 6$.

$g(x,y) = x^2y$

$\nabla f(x,y) = \langle 3, 1 \rangle$

$\nabla g(x,y) = \langle 2xy, x^2 \rangle$

$\nabla f = \lambda \nabla g$

$\Downarrow$

$$\left.\begin{array}{l} 3 = 2xy\lambda \Rightarrow \lambda = \dfrac{3}{2xy} \\ 1 = \lambda x^2 \Rightarrow \lambda = \dfrac{1}{x^2} \end{array}\right\}$$

$x^2y = 6$

Set these $\lambda$'s equal:

$\dfrac{3}{2xy} = \dfrac{1}{x^2}$

$3x^2 = 2xy$ (assumes $x, y \neq 0$)

$3x^2 - 2xy = 0$

$x(3x - 2y) = 0$

$x = 0$

Can't have $x = 0$.

If $x=0$, then $3 = 2xy\lambda$ and $1 = \lambda x^2$ are impossible

$3x - 2y = 0$

$3x = 2y$

$\frac{3}{2}x = y$

$\downarrow$

Put $\frac{3}{2}x = y$ into $x^2y = 6$

$x^2\left(\frac{3}{2}x\right) = 6$

$\dfrac{3x^3}{2} = 6$

$3x^3 = 12$

$x^3 = 4$

$x = \sqrt[3]{4}$

$y = \frac{3}{2}x = \frac{3}{2}\sqrt[3]{4}$

$$\boxed{\text{The numbers are } x = \sqrt[3]{4},\ y = \frac{3\sqrt[3]{4}}{2}.}$$

Ex 2 cont'd:

$y = 2x \Rightarrow \frac{y}{2} = x$

$y = 2z \Rightarrow \frac{y}{2} = z$

$x + y + z = 32$

$x = \frac{y}{2},\ z = \frac{y}{2} \Rightarrow \frac{y}{2} + y + \frac{y}{2} = 32$

$y + y = 32$

$2y = 32$

$y = 16$

$x = \frac{y}{2} = \frac{16}{2} = 8$

$z = \frac{y}{2} = \frac{16}{2} = 8$

Maximum P occurs when $x = 8,\ y = 16,\ z = 8$

Example 4 cont'd:

$$\left.\begin{aligned} xyz &= \lambda(2xy + 2yz) \\ xyz &= \lambda(2yz + xz) \end{aligned}\right\}$$ Set $xyz$'s equal:

$\lambda(2xy + 2yz) = \lambda(2yz + xz)$

$2xy + \cancel{2yz} = \cancel{2yz} + xz$

$2xy = xz$

$2xy - xz = 0$

$x(2y - z) = 0$

$x = 0$ Nope | $2y - z = 0$

$2y = z$

$y = \frac{z}{2}$

From earlier, $y = \frac{x}{2}$

$x = 2y$

$2xy + 2yz + xz = 12$

Put in $x = 2y,\ z = 2y$:

$2(2y)(y) + 2y(2y) + (2y)(2y) = 12$

$4y^2 + 4y^2 + 4y^2 = 12$

$12y^2 = 12$

$y^2 = 1$

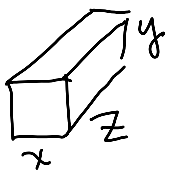
$y = \pm 1$ we want +1.

$x = 2y = 2(1) = 2$

$z = 2y = 2(1) = 2$

So, the box of maximum volume will be 1 m high and have a base of 2m × 2m.

**Example 4:** A rectangular box without a lid is to be made from 12 square meters of cardboard. Find the maximum volume of such a box.




Maximize: Volume $V = xyz$

subject to constraint: $2xy + 2yz + xz = 12$

$g(x,y,z) = 2xy + 2yz + xz$ (need $g(x,y,z) = k$)

$\nabla V(x,y,z) = \lambda \nabla g(x,y,z)$

$\langle yz, xz, xy \rangle = \lambda \langle 2y+z, 2x+2z, 2y+x \rangle$

System of eqns:

$yz = \lambda(2y+z) \xrightarrow{\text{multiply by } x} xyz = \lambda(2xy + xz)$

$xz = \lambda(2x+2z) \xrightarrow{\text{multiply by } y} xyz = \lambda(2xy + 2yz)$

$xy = \lambda(2y+x) \xrightarrow{\text{mult. by } z} xyz = \lambda(2yz + xz)$

$2xy + 2yz + xz = 12$

set xyz's equal: $\Rightarrow \lambda(2xy + xz) = \lambda(2xy + 2yz)$

$\cancel{2xy} + xz = \cancel{2xy} + 2yz$

$xz = 2yz$

$0 = 2yz - xz$

$0 = z(2y - x)$

$z = 0$ Nope | $2y = x$, $y = \frac{x}{2}$

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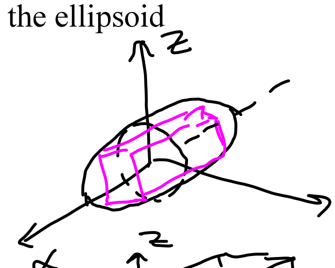
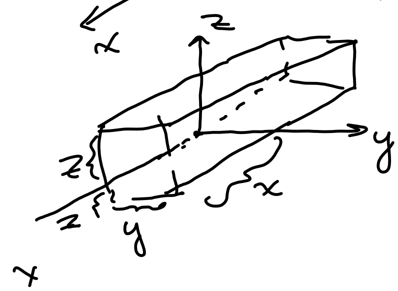
**Example 5:** Find the maximum volume of a rectangular box inscribed in the ellipsoid $x^2 + 3y^2 + 4z^2 = 12$.




The axes will emerge from the centers of the faces of the box. So if the portion in the 1st octant is $xyz$, the total Volume is $8xyz$




$V(x,y,z) = 8xyz$

$g(x,y,z) = x^2 + 3y^2 + 4z^2$

$\nabla V = \langle 8yz, 8xz, 8xy \rangle$

$\nabla g(x,y,z) = \langle 2x, 6y, 8z \rangle$

$\nabla V = \lambda \nabla g \Rightarrow$

$8yz = 2\lambda x \Rightarrow \lambda = \frac{8yz}{2x} = \frac{4yz}{x}$

$8xz = 6y\lambda \Rightarrow \lambda = \frac{8xz}{6y} = \frac{4xz}{3y}$

$8xy = 8z\lambda \Rightarrow \lambda = \frac{8xy}{8z} = \frac{xy}{z}$

$x^2 + 3y^2 + 4z^2 = 12$

$\frac{4yz}{x} = \frac{4xz}{3y}$

$12y^2 z = 4x^2 z$

$12y^2 z - 4x^2 z = 0$

$4z(3y^2 - x^2) = 0$

Throw out $z=0$ | $3y^2 = x^2$

$\lambda = \frac{4xz}{3y}$, $\lambda = \frac{xy}{z}$ $\Rightarrow \frac{4xz}{3y} = \frac{xy}{z}$

$4xz^2 = 3xy^2$

$x(4z^2 - 3y^2) = 0$

$x = 0$ throw out | $4z^2 = 3y^2$, $z^2 = \frac{3}{4}y^2$

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$x^2+3y^2+4z^2=12$

$(3y^2)+3y^2+4\left(\frac{3}{4}y^2\right)=12$

$3y^2+3y^2+3y^2=12 \Rightarrow 9y^2=12$

$y^2=\frac{12}{9}=\frac{4}{3}$

$y=\sqrt{\frac{4}{3}}=\frac{2}{\sqrt{3}}$

$3y^2=x^2$

$3\left(\frac{4}{3}\right)=x^2 \Rightarrow x^2=4$

$x=\pm 2$ choose $x=2$

$z^2=\frac{3}{4}y^2=\frac{3}{4}\left(\frac{4}{3}\right)=1 \Rightarrow z=1$

Volume:

$V=8xyz$

$=8(2)\left(\frac{2}{\sqrt{3}}\right)(1)$

$=\boxed{\frac{32}{\sqrt{3}}}$

**Example 6:** Find the point on the plane $x-y+z=4$ that is closest to the point $(1,2,3)$.

We want to minimize $d=\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$ where $(x,y,z)$ is on the plane.

Minimize $M=d^2$ instead:

$M(x,y,z)=(x-1)^2+(y-2)^2+(z-3)^2$

$g(x,y,z)=x-y+z$

$\nabla M(x,y,z)=\langle 2(x-1),\ 2(y-2),\ 2(z-3)\rangle$

$\nabla g(x,y,z)=\langle 1,-1,1\rangle$

$\nabla M=\lambda\nabla g \Rightarrow$

$2(x-1)=\lambda(1) \Rightarrow \lambda=2x-2 \Rightarrow \lambda+2=2x \Rightarrow \frac{\lambda+2}{2}=x$

$2(y-2)=\lambda(-1) \Rightarrow \lambda=-2(y-2)=-2y+4 \Rightarrow \lambda-4=-2y \Rightarrow y=\frac{\lambda-4}{-2}$

$2(z-3)=\lambda(1) \Rightarrow \lambda=2z-6 \Rightarrow \lambda+6=2z \Rightarrow \frac{\lambda+6}{2}=z$

$x-y+z=4 \Rightarrow \frac{\lambda+2}{2}-\frac{\lambda-4}{-2}+\frac{\lambda+6}{2}=4$

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**Example 7:** Find the extreme values of $f(x,y,z)=x^2y^2z^2$ on the sphere $x^2+y^2+z^2=1$.

$g(x,y,z)=x^2+y^2+z^2$

$\nabla f(x,y,z)=\langle 2xy^2z^2,\ 2x^2yz^2,\ 2x^2y^2z\rangle$

$\nabla g(x,y,z)=\langle 2x,\ 2y,\ 2z\rangle$

$2xy^2z^2=2\lambda x$

$2x^2yz^2=2\lambda y$

$2x^2y^2z=2\lambda z$

$x^2+y^2+z^2=1$

Divide each of these by 2, and then multiply by $x$, $y$, or $z$ to get $x^2y^2z^2$ on left side

$x^2y^2z^2=\lambda x^2$

$x^2y^2z^2=\lambda y^2$

$x^2y^2z^2=\lambda z^2$

$\Rightarrow \lambda x^2=\lambda y^2=\lambda z^2$

Assuming $\lambda\neq 0$, this means that

$x^2=y^2=z^2$

Constraint: $x^2+y^2+z^2=1$

$x^2+x^2+x^2=1$

$3x^2=1$

$x^2=\frac{1}{3}$

$x=\pm\frac{1}{\sqrt{3}}$

$x=\pm\frac{1}{\sqrt{3}} \Rightarrow y=\pm\frac{1}{\sqrt{3}},\ z=\pm\frac{1}{\sqrt{3}}$

In this case, $f(x,y,z)=x^2y^2z^2=\left(\pm\frac{1}{\sqrt{3}}\right)^2\left(\pm\frac{1}{\sqrt{3}}\right)^2\left(\pm\frac{1}{\sqrt{3}}\right)^2=\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{27}$

Earlier we assumed $\lambda\neq 0$. If $\lambda=0$, then at least one of $x$, $y$, and $z$ must be 0. In that case, $f(x,y,z)=x^2y^2z^2=0$ also.

Minimum value: 0

Maximum Value: $\frac{1}{27}$

Ex 6 cont'd:

$$\frac{\lambda+2}{2} - \frac{\lambda-4}{-2} + \frac{\lambda+6}{2} = 4$$

Multiply by 2: $\lambda+2+\lambda-4+\lambda+6 = 8$

$$3\lambda + 4 = 8$$

$$3\lambda = 4$$

$$\lambda = \frac{4}{3}$$

$$x = \frac{\lambda+2}{2} = \frac{\frac{4}{3}+2}{2} = \frac{\frac{10}{3}}{2} = \frac{10}{6} = \frac{5}{3}$$

$$y = \frac{\lambda-4}{-2} = \frac{\frac{4}{3}-4}{-2} = \frac{-8/3}{-2} = \frac{8}{6} = \frac{4}{3}$$

$$z = \frac{\lambda+6}{2} = \frac{\frac{4}{3}+6}{2} = \frac{22/3}{2} = \frac{22}{6} = \frac{11}{3}$$

$\boxed{\text{The closest point is } \left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)}$

**Optimization problems with two constraints:**

Suppose we want to find the extrema of $f(x,y,z)$ subject to <u>two</u> constraints, $g(x,y,z)=k$ and $h(x,y,z)=c$. Then, the gradient of *f* must be a linear combination of the gradients of *g* and *h*. In other words, if *f* has an extremum at $(x_0,y_0,z_0)$, then

$\mu$ = mu

$$\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)+\mu\nabla h(x_0,y_0,z_0)$$, where $\lambda$ and $\mu$ are scalars.

Lagrange's method results in five equations in five variables:

$$f_x=\lambda g_x+\mu h_x$$
$$f_y=\lambda g_y+\mu h_y$$
$$f_z=\lambda g_z+\mu h_z$$
$$g(x,y,z)=k$$
$$h(x,y,z)=c$$

**<u>Example 8:</u>** Find the extreme values of $f(x,y,z)=3x-y-3z$ on the curve of intersection of $x+y-z=0$ and $x^2+2z^2=1$.

$f(x,y,z)=3x-y-3z$
$\nabla f(x,y,z)=\langle 3,-1,-3\rangle$

$g(x,y,z)=x+y-z$
$\nabla g(x,y,z)=\langle 1,1,-1\rangle$
$h(x,y,z)=x^2+2z^2$
$\nabla h(x,y,z)=\langle 2x,0,4z\rangle$

$\nabla f=\lambda\nabla g+\mu\nabla h$

$3=1\lambda+2x\mu$
$-1=1\lambda+0 \implies \lambda=-1$
$-3=-1\lambda+4\mu z$
$x+y-z=0$
$x^2+2z^2=1$

$3=-1+2x\mu$
$4=2x\mu$
$\frac{4}{2\mu}=x$
$\frac{2}{\mu}=x$

$-3=-1(-1)+4\mu z$
$-3=1+4\mu z$
$-4=4\mu z$
$\frac{-4}{4\mu}=z$
$-\frac{1}{\mu}=z$

2nd constraint:
$x^2+2z^2=1$
$\left(\frac{2}{\mu}\right)^2+2\left(-\frac{1}{\mu}\right)^2=1$
$\frac{4}{\mu^2}+\frac{2}{\mu^2}=1$
$\frac{6}{\mu^2}=1$
$6=\mu^2$
$\pm\sqrt{6}=\mu$

$\boxed{\mu=\sqrt{6}}$ $x=\frac{2}{\mu}=\frac{2}{\sqrt{6}}$
$z=-\frac{1}{\mu}=-\frac{1}{\sqrt{6}}$
1st constraint: $x+y-z=0$
$\frac{2}{\sqrt{6}}+y-\left(-\frac{1}{\sqrt{6}}\right)=0$
$\frac{3}{\sqrt{6}}+y=0$
$y=-\frac{3}{\sqrt{6}}$

$\boxed{\mu=-\sqrt{6}}$ $x=\frac{2}{\mu}=\frac{2}{-\sqrt{6}}$
$z=-\frac{1}{\mu}=-\frac{1}{-\sqrt{6}}=\frac{1}{\sqrt{6}}$
$x+y-z=0$
$-\frac{2}{\sqrt{6}}+y-\frac{1}{\sqrt{6}}=0$
$-\frac{3}{\sqrt{6}}+y=0$
$y=\frac{3}{\sqrt{6}}$

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$f\left(\frac{2}{\sqrt6}, -\frac{3}{\sqrt6}, -\frac{1}{\sqrt6}\right) = 3\left(\frac{2}{\sqrt6}\right) - \left(-\frac{3}{\sqrt6}\right) - 3\left(-\frac{1}{\sqrt6}\right) = \frac{6}{\sqrt6} + \frac{3}{\sqrt6} + \frac{3}{\sqrt6} = \frac{12}{\sqrt6} = \frac{12\sqrt6}{6} = 2\sqrt6$

$\boxed{f\left(-\frac{2}{\sqrt6}, \frac{3}{\sqrt6}, \frac{1}{\sqrt6}\right) = -2\sqrt6 \text{ Minimum}}$



$\boxed{\text{Maximum: } f\left(\frac{2}{\sqrt6}, -\frac{3}{\sqrt6}, -\frac{1}{\sqrt6}\right) = 2\sqrt6}$

**Example 9:** Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2z = 6$ and $x + y = 12$.

$g(x,y,z) = x + 2z$

$h(x,y,z) = x + y$

$\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$

$\nabla g(x,y,z) = \langle 1, 0, 2 \rangle$

$\nabla h(x,y,z) = \langle 1, 1, 0 \rangle$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$2x = \lambda + \mu$

$2y = \lambda(0) + \mu \Rightarrow \mu = 2y$ } then use

$2z = 2\lambda + 0(\mu) \Rightarrow \lambda = z$

$x + 2z = 6 \Rightarrow 2z = 6 - x \Rightarrow z = \frac{6-x}{2}$

$x + y = 12 \Rightarrow y = 12 - x$

$2x = \lambda + \mu$

$2x = z + 2y$

$\rightarrow 2x = \frac{6-x}{2} + 2(12-x)$

$4x = 6 - x + 4(12-x)$

$4x = 6 - x + 48 - 4x$

$4x = 54 - 5x$

$9x = 54$

$x = 6$

$y = 12 - x$
$= 12 - 6 = 6$

$z = \frac{6-x}{2} = \frac{6-6}{2} = \frac{0}{2} 0$

$\boxed{\text{Minimum is } f(6,6,0) = 72}$

Note: $f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = f_{yx} = 0$

This is a relative min.

**Example 10:** Suppose the temperature at each point on the sphere $x^2 + y^2 + z^2 = 50$ is given by the function $T(x, y, z) = 100 + x^2 + y^2$. Find the maximum temperature on the curve formed by the intersection of the sphere and the plane $x - z = 0$.

$g(x,y,z) = x^2 + y^2 + z^2$

$h(x,y,z) = x - z$

$\nabla T(x,y,z) = \langle 2x, 2y, 0 \rangle$

$\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$

$\nabla h(x,y,z) = \langle 1, 0, -1 \rangle$

$\nabla T = \lambda \nabla g + \mu \nabla h$

$2x = \lambda(2x) + \mu$

$2y = \lambda(2y) + 0 \Rightarrow \lambda = \frac{2y}{2y} = 1$, or else $y = 0$

$0 = \lambda(2z) - \mu$

$x^2 + y^2 + z^2 = 50$

$x - z = 0$

For $\lambda = 1$

$2x = 2x\lambda + \mu$

$2x = 2x(1) + \mu$

$0 = \mu$

$0 = 2\lambda z - \mu$

$0 = 2(1)z - 0$

$0 = 2z$

$0 = z$

Then $x - z = 0$ gives us $x - 0 = 0$

$x = 0$

$x^2 + y^2 + z^2 = 50$

$0^2 + y^2 + 0^2 = 50$

$y^2 = 50$

$y = \pm\sqrt{50} = \pm 5\sqrt2$

For $y = 0$

$x^2 + y^2 + z^2 = 50$

$x^2 + 0^2 + z^2 = 50$

$x^2 + z^2 = 50$

$x - z = 0 \Rightarrow x = z$

$x^2 + x^2 = 50$

$2x^2 = 50$

$x^2 = 25$

$x = \pm 5$

$x = 5 \Rightarrow z = 5$

$x = -5 \Rightarrow z = -5$

So, our points are: $(0, 5\sqrt2, 0)$, $(0, -5\sqrt2, 0)$, $(5, 0, 5)$, $(-5, 0, -5)$

$T(0, 5\sqrt2, 0) = 100 + 0^2 + (\sqrt{50})^2 = 50$ min

$T(0, -5\sqrt2, 0) = 100 + 0^2 + (-\sqrt{50})^2 = 50$ min

$T(5, 0, 5) = 100 + 5^2 + 0^2 = 125$

$T(-5, 0, -5) = 100 + (-5)^2 + 0^2 = 125$

$\boxed{\text{Maximum Temperature is } T(5,0,5) = T(-5,0,-5) = 125}$

# Homework Qs

**13.10 # 5** $f(x,y) = 2x + 2xy + y$, subject to $2x + y = 100$

$g(x,y) = 2x + y$

$\nabla g(x,y) = \langle 2, 1 \rangle$

$\nabla f(x) = \langle 2 + 2y, 2x + 1 \rangle$

$$\left.\begin{aligned} 2 + 2y &= 2\lambda \\ 2x + 1 &= \lambda \\ 2x + y &= 100 \end{aligned}\right\}$$

$2 + 2y = 2\lambda$

$\hookrightarrow 2y = 2\lambda - 2$

$y = \lambda - 1$

$2x + 1 = \lambda$

$\hookrightarrow 2x = \lambda - 1$

$x = \frac{\lambda - 1}{2}$

Put $y = \lambda - 1$, $x = \frac{\lambda - 1}{2}$ into $2x + y = 100$

$2\left(\frac{\lambda - 1}{2}\right) + \lambda - 1 = 100$

$\lambda - 1 + \lambda - 1 = 100$

$2\lambda - 2 = 100$

$2\lambda = 102$

$\lambda = 51$

$y = \lambda - 1 \Rightarrow y = 51 - 1 = 50$

$x = \frac{\lambda - 1}{2} \Rightarrow x = \frac{51 - 1}{2} = \frac{50}{2} = 25$