13.1: Introduction to Functions of Several Variables

You are already acquainted with functions of several variables, even if you haven't written them in function notation:

Volume of a rectangular solid: $V(l,w,h) = lwh$

Volume of a right circular cone: $V(r,h) = \frac{1}{3}\pi r^2 h$

Example 1: Suppose $z = f(x,y) = x^3 + 2xy + 3y^2$. Evaluate $f(-2,3)$.

Example 2: Suppose $g(x,y,z) = 2xz^2 - 3y^3 + 5y^2 z$. Evaluate $g(3,4,-2)$.

Domain and range of functions of several variables:

The *domain* of a function of *n* variables is the set of points (inputs) in $\mathbb{R}^n$ for which the function results in a valid output. The *range* of a function is the set of all outputs of the function.

Note: The graph of a function of *n* variables is a set of points in $\mathbb{R}^{n+1}$. (When we combine the output of the function with the values of all the input variables, we add a dimension.)

For example, the graph of a function of one variable is a curve in $\mathbb{R}^2$. If we start with $f(x)$, we can let $y = f(x)$, and then the graph consists of ordered pairs (x,y).

Similarly, the graph of a function of two variables is a curve in $\mathbb{R}^3$. If we start with $f(x,y)$, we can let $z = f(x,y)$, and then the graph consists of ordered triples (x,y,z).

To answer the difficulty in writing a clear definition of a tangent line, we can define it as the limiting position of the secant line as the second point approaches the first.

Example 3: Find the domain and range of $f(x,y) = \frac{xy}{x-y}$.

Example 4: Find the domain and range of $f(x,y) = \ln(xy - 6)$.

Example 5: Find the domain and range of $f(x,y) = \ln(xy - 6)$.

Example 6: Find the domain and range of $f(x,y) = \sqrt{4 - x^2 - 9y^2}$.

Example 7: Sketch the graph of the function $f(x,y) = \frac{x^2}{9} + \frac{y^2}{16}$.

Example 8: Sketch the graph of the function $f(x,y) = \sqrt{1 + x^2 + y^2}$.

Level curves:

A *level curve* for a function of two variables is a set of points in $\mathbb{R}^2$ for which the function value (output) is constant.

For example, if $z = f(x, y)$, then the level curve for $z = 1$ is the set of points (x, y) for which $z = f(x, y) = 1$. Setting $z = 1$ in the equation of the function produces an equation in *x* and *y* only. The graph of this equation is the level curve for $z = 1$. If we draw the level curves for $z = 1$, $z = 2$, $z = 3$, etc. in the *xy*-plane, they'll help us visualize the graph of the function. A drawing of level curves is called a *contour map*.

Recall: The intersection of a surface in $\mathbb{R}^3$ with a plane is called the *trace* of that surface in the plane. So, for a curve in which $z = f(x, y)$, the level curve for $z = c$ is just the trace of the surface in the plane $z = c$.

Note: In order for a contour map to be helpful in visualization, the *z*-values for the level curves should be equally spaced. Then, level curves that are far apart indicate than *z* is changing slowly. Level curves very close together indicate a rapid change in *z*.

Example 9: Draw several level curves for $f(x, y) = x^2 - y^2$.

Example 10: Draw several level curves for $f(x, y) = x - y^2$.

Level surfaces:

When we extend the notion of level curves to functions of three variables, we get *level surfaces*. A level surface for the function $f(x, y, z)$ is the set of points in $\mathbb{R}^3$ for which $f(x, y, z) = k$ (k a constant).

<u>Example 11:</u> Consider the function $f(x, y, z) = x^2 + y^2 + z^2$. What do the level surfaces look like?