13.1: Introduction to Functions of Several Variables

You are already acquainted with functions of several variables, even if you haven't written them in function notation:

Volume of a rectangular solid: $V(l,w,h) = lwh$

Volume of a right circular cone: $V(r,h) = \frac{1}{3}\pi r^2 h$

Example 1: Suppose $z = f(x,y) = x^3 + 2xy + 3y^2$. Evaluate $f(-2,3)$.

$$f(-2,3) = (-2)^3 + 2(-2)(3) + 3(3)^2$$
$$= -8 - 12 + 27$$
$$= -20 + 27 = \boxed{7}$$

Example 2: Suppose $g(x,y,z) = 2xz^2 - 3y^3 + 5y^2 z$. Evaluate $g(3,4,-2)$.

$$g(3,4,-2) = 2(3)(-2)^2 - 3(4)^3 + 5(4)^2(-2)$$
$$= 24 - 192 - 160 \quad = 24 - 352$$
$$= -128$$

Domain and range of functions of several variables:

The *domain* of a function of *n* variables is the set of points (inputs) in $\mathbb{R}^n$ for which the function results in a valid output. The *range* of a function is the set of all outputs of the function.

Note: The graph of a function of *n* variables is a set of points in $\mathbb{R}^{n+1}$. (When we combine the output of the function with the values of all the input variables, we add a dimension.)

For example, the graph of a function of one variable is a curve in $\mathbb{R}^2$. If we start with $f(x)$, we can let $y = f(x)$, and then the graph consists of ordered pairs (x, y).

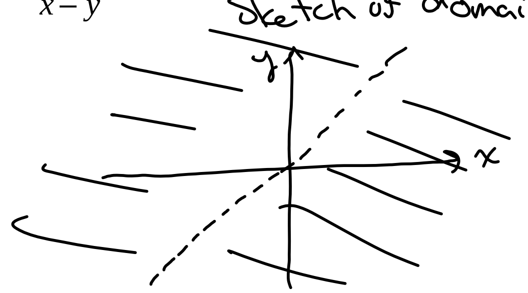
Similarly, the graph of a function of two variables is a curve in $\mathbb{R}^3$. If we start with $f(x,y)$, we can let $z = f(x,y)$, and then the graph consists of ordered triples (x, y, z).

~~To answer the difficulty in writing a clear definition of a tangent line, we can define it as the limiting position of the secant line as the second point approaches the first.~~

Example 3: Find the domain and range of $f(x,y) = \frac{xy}{x-y}$.

Cannot have denominator $= 0$, so $x \neq y$

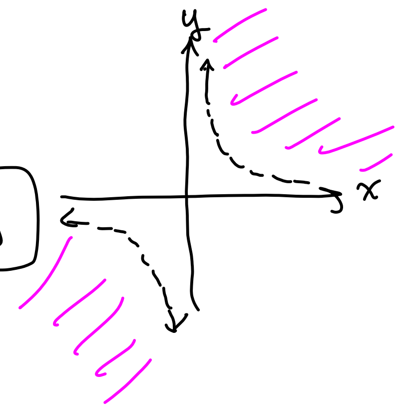
Sketch of domain:



Domain: $\{(x,y) \mid x \neq y\}$

Range: $(-\infty, \infty)$

Graph $y = \frac{6}{x}$



Example 4: Find the domain and range of $f(x, y) = \ln(xy - 6)$.

Can only apply ln function to numbers > 0.

So, $xy > 6$

For $x > 0$, $y > \frac{6}{x}$

For $x < 0$, $y < \frac{6}{x}$

$\boxed{\text{Domain: } \{(x,y) \mid xy > 6\}}$

Range: $(-\infty, \infty)$

Example 5: Find the domain and range of $f(x, y) = \ln(xy - 6)$.

Example 6: Find the domain and range of $f(x, y) = \sqrt{4 - x^2 - 9y^2}$.

Domain:
$\{(x,y) \mid x^2 + 9y^2 \le 4\}$
(all points inside or on the ellipse)

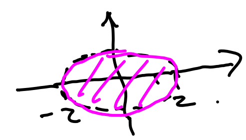
Range:
$[0, \infty)$

$4 - x^2 - 9y^2 \ge 0$

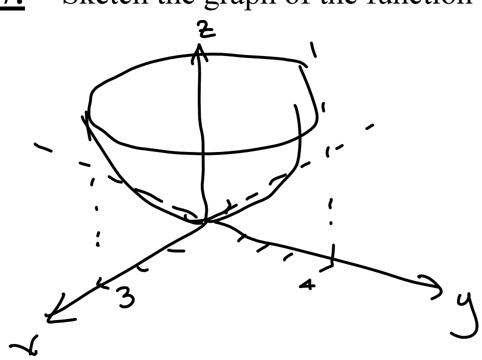
$4 \ge x^2 + 9y^2$

$x^2 + 9y^2 \le 4$

$\frac{x^2}{4} + \frac{y^2}{4/9} \le 1$



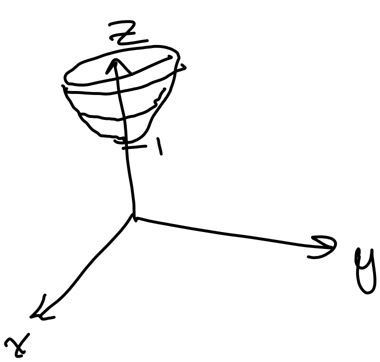
Example 7: Sketch the graph of the function $f(x, y) = \frac{x^2}{9} + \frac{y^2}{16}$.



$z = \frac{x^2}{9} + \frac{y^2}{16}$

Range: $[0, \infty)$

Example 8: Sketch the graph of the function $f(x, y) = \sqrt{1 + x^2 + y^2}$.



$z = f(x,y) \Rightarrow z = \sqrt{1 + x^2 + y^2}$

$z^2 = 1 + x^2 + y^2$

$z^2 - x^2 - y^2 = 1$

Traces:

xy: $z = 0 \Rightarrow$ impossible

xz: $z^2 - x^2 = 1$ hyperbola

yz: $z^2 - y^2 = 1$

Level curves:

A *level curve* for a function of two variables is a set of points in $\mathbb{R}^2$ for which the function value (output) is constant.

For example, if $z = f(x, y)$, then the level curve for $z = 1$ is the set of points (x, y) for which $z = f(x, y) = 1$. Setting $z = 1$ in the equation of the function produces an equation in *x* and *y* only. The graph of this equation is the level curve for $z = 1$. If we draw the level curves for $z = 1$, $z = 2$, $z = 3$, etc. in the *xy*-plane, they'll help us visualize the graph of the function. A drawing of level curves is called a *contour map*.

Recall: The intersection of a surface in $\mathbb{R}^3$ with a plane is called the *trace* of that surface in the plane. So, for a curve in which $z = f(x, y)$, the level curve for $z = c$ is just the trace of the surface in the plane $z = c$.

Note: In order for a contour map to be helpful in visualization, the *z*-values for the level curves should be equally spaced. Then, level curves that are far apart indicate than *z* is changing slowly. Level curves very close together indicate a rapid change in *z*.

$k = \sqrt{16-x^2-y^2}$

$k^2 = 16-x^2-y^2$

$x^2+y^2 = 16-k^2$

Example 9: Draw several level curves for $f(x,y) = \sqrt{16-x^2-y^2}$

$z = f(x,y) = \sqrt{16-x^2-y^2}$

Choose several different values for z.

$z=0 \Rightarrow 0 = \sqrt{16-x^2-y^2}$

$0 = 16-x^2-y^2$

$x^2+y^2 = 16$

$z=1 \Rightarrow 1 = \sqrt{16-x^2-y^2}$

$x^2+y^2 = 16-1^2$

$x^2+y^2 = 15$

$z=2 \Rightarrow x^2+y^2 = 16-2^2$

$x^2+y^2 = 12$

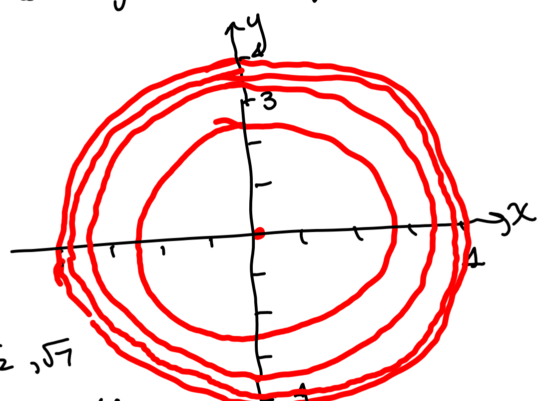
$z=3 \Rightarrow x^2+y^2 = 16-3^2$

$x^2+y^2 = 7$

Circles of radii $4, \sqrt{15}, \sqrt{12}, \sqrt{7}$

$\approx 4, 3.87, 3.46, 2.65$

$z=4 \Rightarrow x^2+y^2 = 16-4^2 = 0$



Example 10: Draw several level curves for $f(x, y) = x - y^2$.

$z=k \Rightarrow k = x - y^2$

$y^2+k = x$

Right-opening parabola shifted horizontally by k units

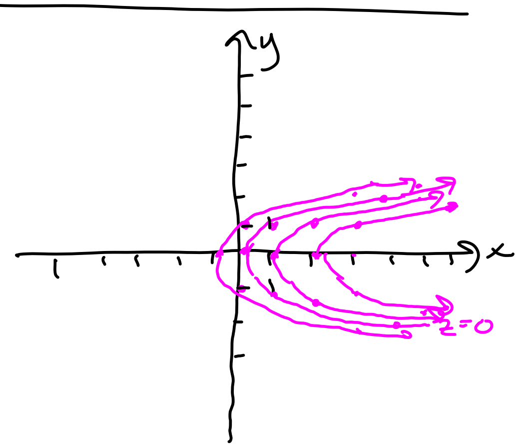
$k=0 \Rightarrow y^2 = x$

$k=1 \Rightarrow y^2+1 = x$

$k=2 \Rightarrow y^2+2 = x$

$k=-1 \Rightarrow y^2-1 = x$

y	x=y²
0	0
±1	1
±2	4



Level surfaces:

When we extend the notion of level curves to functions of three variables, we get *level surfaces*. A level surface for the function $f(x, y, z)$ is the set of points in $\mathbb{R}^3$ for which $f(x, y, z) = k$ (k a constant).

Example 11: Consider the function $f(x, y, z) = x^2 + y^2 + z^2$. What do the level surfaces look like?

$w = f(x,y,z) \Rightarrow \quad w = x^2 + y^2 + z^2$

$w = 0 \Rightarrow \quad 0 = x^2 + y^2 + z^2$ Pt $(0,0,0)$

$w = 1 \Rightarrow \quad 1 = x^2 + y^2 + z^2$ Sphere of radius 1

$w = 2 \Rightarrow \quad 2 = x^2 + y^2 + z^2 \Rightarrow$ sphere of radius $\sqrt{2}$

$w = 3 \Rightarrow \quad 3 = x^2 + y^2 + z^2$ Sphere radius $\sqrt{3}$

$w = 4 \Rightarrow \quad 4 = x^2 + y^2 + z^2$ Sphere of radius 2

Level surface for $w = 4$:
(radius = 2)