13.2: Limits and Continuity

We will skip the 3-dimensional version of the ε - δ definition of a limit.

Main principle to remember for determining whether $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists:

If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ is to be true, then the values of $f(x,y)$ must approach L as (x,y) approaches (a,b) regardless of path. (No matter what path (x,y) follows when approaching (a,b), the function values still approach L.)

In other words, if $f(x,y)\to L_1$ as $(x,y)\to(a,b)$ along the curve C_1, but $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ along the curve C_2, and $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 1: Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$, if it exists.

$\frac{0}{0}$ indeterminate form

1st, approach (0,0) along x-axis. (So y=0)

As $(x,0)\to(0,0)$, $f(x,0) = \dfrac{x(0)}{x^2+0^2} = \dfrac{0}{x^2} = 0$. So, as $(x,0)\to(0,0)$, $\dfrac{xy}{x^2+y^2}\to 0$

2nd, approach (0,0) along y-axis (so x=0)

As $(0,y)\to(0,0)$, $z = f(0,y) = \dfrac{0(y)}{0^2+y^2} = \dfrac{0}{y^2} = 0$. So, as $(0,y)\to(0,0)$, $\dfrac{xy}{x^2+y^2}\to 0$

3rd, approach (0,0) along the line $y=x$. (So can write $y=x$)

As $(x,y)=(x,x)\to(0,0)$, $z = f(x,x) = \dfrac{x(x)}{x^2+x^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2}$

don't match!

$\boxed{\text{Limit does not exist}}$

Example 2: Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$, if it exists.

Approach along $x=0$: $f(0,y) = \dfrac{0(y^2)}{0^2+y^4} = \dfrac{0}{y^4} = 0$

As $(0,y)\to(0,0)$,

Approach along $y=0$: as $(x,0)\to(0,0)$, $f(x,0)\to \dfrac{x(0)^2}{x^2+0^4} = \dfrac{0}{x^2} = 0$

Along the line $y=mx$:

As $(x,mx)\to(0,0)$, $f(x,mx) = \dfrac{x(mx)^2}{x^2+(mx)^4} = \dfrac{m^2x^3}{x^2+m^4x^4} = \dfrac{m^2x^3}{x^2(1+m^4x^2)}$

$= \dfrac{m^2x}{1+m^4x^2} \to \dfrac{m^2(0)}{1+m^4(0)^2} = \dfrac{0}{1} = 0$

Along the parabola $x=y^2$:

As $(y^2,y)\to(0,0)$, $f(y^2,y) = \dfrac{y^2y^2}{(y^2)^2+y^4} = \dfrac{y^4}{y^4+y^4} = \dfrac{y^4}{2y^4} = \dfrac{1}{2}$

$\boxed{\text{Limit doesn't exist.}}$

Example 3: Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$, if it exists.

Along x-axis: $(y=0)$

As $(x,0) \to (0,0)$, $f(x,0) \to \frac{3x^2(0)}{x^2+0^2} = \frac{0}{x^2} = 0$

Along y-axis. $(x=0)$

As $(0,y) \to (0,0)$, $f(0,y) \to \frac{3(0)^2 y}{0^2+y^2} = \frac{0}{y^2} = 0$

Along $y=mx$:

As $(x,mx) \to (0,0)$, $f(x,mx) = \frac{3x^2(mx)}{x^2+(mx)^2} = \frac{3mx^3}{x^2+m^2x^2} = \frac{3mx^3}{x^2(1+m)} = \frac{3mx}{1+m} \to \frac{0}{1+m} = 0$

Along $y=kx^2$

As $(x,kx^2) \to (0,0)$, $f(x,kx^2) = \frac{3x^2(kx^2)}{x^2+(kx^2)^2} = \frac{3kx^4}{x^2+k^2x^4} = \frac{3kx^4}{x^2(1+k^2x^2)} = \frac{3kx^2}{1+k^2x^2} \to \frac{3k(0)^2}{1+0} = 0$

$\boxed{\text{The limit is } 0}$

Example 4: Find $\lim_{(x,y)\to(2,1)} \frac{xy}{x^2+y^2}$, if it exists.

$\lim_{(x,y)\to(2,1)} \frac{xy}{x^2+y^2} = \frac{2(1)}{2^2+1^2} = \boxed{\frac{2}{5}}$

Continuity:

> <u>Definition</u>: A function f of two variables is continuous at a point (a,b) in an open region R if
>
> $$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) \;.$$
>
> The function is continuous on the open region R if it is continuous at every point in R.

In general, if we combine continuous functions using addition, multiplication, composition, or division, the combined functions are continuous at the places they are defined. (When dividing functions, we may introduce discontinuities due to 0 denominators.)

Example 5: Discuss the continuity of $f(x,y) = \frac{xy}{x^2+y^2}$.

Not continuous at $(0,0)$

Example 6: Discuss the continuity of $f(x,y,z) = \frac{\sqrt{y}}{x^2-y^2+z^2}$.

Defined for $y \geq 0$

$x^2-y^2+z^2 \neq 0$

$z^2 = y^2 - x^2$

$x^2+z^2 = y^2$ Continuous except where $z^2 = y^2 - x^2$