<u>13.5: Partial Derivatives</u>

Consider a function of two or more variables, such as $f(x,y)=x^3+2x^3y^3-y^5$. What would the derivative represent? Rate of change with respect to what?

Partial differentiation is the process of finding the rate of change in a function with respect to one variable, while holding the other variables constant.

<u>Definition</u>: Suppose $z=f(x,y)$ is a function of *x* and *y*.

The partial derivative of *f* (or *z*) with respect to *x* is

$$f_x(x,y)=\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x},\ \text{provided this limit exists.}$$

The partial derivative of *f* (or *z*) with respect to *y* is

$$f_y(x,y)=\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y},\ \text{provided this limit exists.}$$

The partial derivative with respect to *x*, $f_x(x,y)$, gives the slope of the surface in the direction of the *x*-axis.

The partial derivative with respect to *y*, $f_y(x,y)$, gives the slope of the surface in the direction of the *y*-axis.

(We often refer to these as the first partial derivatives, to distinguish them from the second and higher-order partial derivatives.)

<u>Example 1:</u> Find the first partial derivatives of $f(x,y)=x^3+2x^3y^3-y^5$.

Example 2: Suppose $z = 9 - x^2 - y^2$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(\sqrt{3}, 5)$.

Example 3: Suppose $z = \ln \sqrt{xy}$. Find all the first partial derivatives.

Example 4: Suppose $f(x, y) = \cos(x^2 + y^2)$. Find all the first partial derivatives.

Example 5: Suppose $z = x^2 \sqrt{1 + xy}$. Find all the first partial derivatives.

Example 6: Suppose $f(x, y) = \frac{xy}{x^2 + y^2}$. Find all the first partial derivatives.

Example 7: Suppose $f(x,y,z)=3xyz^2+\dfrac{1}{xy}-2z$. Find all the first partial derivatives.

Example 8: Suppose $z=x^2e^{xy^2}$. Find all the first partial derivatives.

Example 9: Suppose $f(x,y,z)=x^2y^3+2xyz-3yz$. Find all the first partial derivatives at the point $(-2,1,2)$.

Example 10: Find the slope in the *x*- and *y*-directions of the surface given by $f(x,y)=x\sin(x+y)$ at the point $\left(\dfrac{\pi}{2},\dfrac{\pi}{3}\right)$.

Higher-order partial derivatives:

The second partial derivatives of $z = f(x, y)$ are defined as follows:

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

<u>**Example 11:**</u> Suppose $f(x, y) = x\sin(4x - 3y)$. Find all the second partial derivatives.

<u>**Example 12:**</u> Suppose $f(x, y, z) = \dfrac{3z^2}{x + 2y}$. Find all the second partial derivatives.

<u>Theorem</u>: <u>Equality of Mixed Partial Derivatives</u>
(sometimes known as Clairut's Theorem).

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then

$$f_{xy}(x,y) = f_{yx}(x,y) \text{ for every } (x,y) \text{ in } R.$$

This theorem also applies to third- and higher order derivatives, and to functions of three or more variables. As long as all the higher-order partial derivatives are continuous, all the mixed partial derivatives of that order will be equal.

<u>Example 13:</u> Suppose $w(x,y,z) = e^{x^2yz}$. Find $w_{xyz}(x,y,z)$, $w_{zyx}(x,y,z)$, $w_{xyx}(x,y,z)$, and $w_{xxy}(x,y,z)$.

<u>Example 14:</u> Determine whether the following functions satisfy the partial differential equation (PDE) $u_{xx} + u_{yy} = 0$, known as Laplace's equation.

a) $u = x^3 + 3xy^2$

b) $u = e^{-x}\cos y - e^{-y}\cos x$