~~13.5.1~~
13.3.1

13.3

~~13.5:~~ Partial Derivatives

Consider a function of two or more variables, such as $f(x,y)=x^3+2x^3y^3-y^5$. What would the derivative represent? Rate of change with respect to what?

Partial differentiation is the process of finding the rate of change in a function with respect to one variable, while holding the other variables constant.

Definition: Suppose $z=f(x,y)$ is a function of x and y.

The partial derivative of f (or z) with respect to x is

$f_x(x,y)$ $=\frac{\partial z}{\partial x}$

$$f_x(x,y)=\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x},$$ provided this limit exists.

The partial derivative of f (or z) with respect to y is

$$f_y(x,y)=\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y},$$ provided this limit exists.

The partial derivative with respect to x, $f_x(x,y)$, gives the slope of the surface in the direction of the x-axis.

The partial derivative with respect to y, $f_y(x,y)$, gives the slope of the surface in the direction of the y-axis.

(We often refer to these as the first partial derivatives, to distinguish them from the second and higher-order partial derivatives.)

Example 1: Find the first partial derivatives of $f(x,y)=x^3+2x^3y^3-y^5$.

$f(x,y)=x^3+2x^3y^3-y^5$

$f_x(x,y)=3x^2+2y^3(3x^2)+0=\boxed{3x^2+6x^2y^3}$

$f_y(x,y)=0+2x^3(3y^2)-5y^4=\boxed{6x^3y^2-5y^4}$

Example 2: Suppose $z = 9 - x^2 - y^2$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(\sqrt{3}, 5)$.

$$\frac{\partial z}{\partial x} = 0 - 2x - 0 = -2x$$

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=\sqrt{3}\\ y=5}} = \boxed{-2\sqrt{3}}$$

$$\frac{\partial z}{\partial y} = 0 - 0 - 2y = -2y$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=\sqrt{3}\\ y=5}} = -2(5) = \boxed{-10}$$

Example 3: Suppose $z = \ln\sqrt{xy}$. Find all the first partial derivatives.

$$z = \ln (xy)^{1/2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{(xy)^{1/2}} \frac{\partial}{\partial x}(xy)^{1/2} = \frac{1}{\sqrt{xy}}\left(\frac{1}{2}\right)(xy)^{-1/2}\frac{\partial}{\partial x}(xy) = \frac{1}{2\sqrt{xy}}(xy)^{-1/2}(y)$$

$$= \frac{1}{2\sqrt{xy}}\left(\frac{1}{\sqrt{xy}}\right)y = \frac{y}{2xy} = \boxed{\frac{1}{2x}}$$

$$z = \ln (xy)^{1/2} = \frac{1}{2}\ln(xy)$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\cdot\frac{1}{xy}\frac{\partial}{\partial y}(xy) = \frac{1}{2xy}(x) = \boxed{\frac{1}{2y}}$$

Example 4: Suppose $f(x, y) = \cos(x^2 + y^2)$. Find all the first partial derivatives.

$$f_x(x,y) = -\sin(x^2+y^2)\frac{\partial}{\partial x}(x^2+y^2)$$

$$= -(\sin(x^2+y^2))(2x+0)$$

$$= \boxed{-2x\sin(x^2+y^2)}$$

$$f_y(x,y) = -\sin(x^2+y^2)\frac{\partial}{\partial y}(x^2+y^2)$$

$$= (-\sin(x^2+y^2))(0+2y)$$

$$= \boxed{-2y\sin(x^2+y^2)}$$

Product Rule

Example 5: Suppose $z = x^2\sqrt{1+xy}$. Find all the first partial derivatives.

$$z = x^2(1+xy)^{1/2}$$

$$\frac{\partial z}{\partial x} = x^2\frac{\partial}{\partial x}(1+xy)^{1/2} + (1+xy)^{1/2}\frac{\partial}{\partial x}(x^2)$$

$$= x^2\left(\frac{1}{2}\right)(1+xy)^{-1/2}(0+y) + (1+xy)^{1/2}(2x)$$

$$= \frac{x^2y}{2\sqrt{1+xy}} + 2x\sqrt{1+xy}$$

$$\frac{x^2y}{2\sqrt{1+xy}} + \frac{2x\sqrt{1+xy}}{1}\left(\frac{\sqrt{1+xy}}{\sqrt{1+xy}}\right)\left(\frac{2}{2}\right)$$

$$= \frac{x^2y + 4x(1+xy)}{2\sqrt{1+xy}}$$

$$= \frac{x^2y + 4x + 4x^2y}{2\sqrt{1+xy}} = \boxed{\frac{5x^2y+4x}{2\sqrt{1+xy}}}$$

$$z = x^2(1+xy)^{1/2}$$

$$\frac{\partial z}{\partial y} = x^2\left(\frac{1}{2}\right)(1+xy)^{-1/2}(0+x)$$

$$= \boxed{\frac{x^3}{2\sqrt{1+xy}}}$$ ← Ex 5

Example 6: Suppose $f(x, y) = \frac{xy}{x^2 + y^2}$. Find all the first partial derivatives.

Quotient Rule for both

$$f_x(x,y) = \frac{(x^2+y^2)\frac{\partial}{\partial x}(xy) - xy\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(y) - xy(2x+0)}{(x^2+y^2)^2}$$

$$= \frac{x^2y + y^3 - 2x^2y}{(x^2+y^2)^2} = \boxed{\frac{y^3 - x^2y}{(x^2+y^2)^2}}$$

See next page

Ex 6 cont'd:

$$f_y(x,y) = \frac{(x^2+y^2)\frac{\partial}{\partial y}(xy) - xy\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} = \frac{(x^2+y^2)(x) - xy(0+2y)}{(x^2+y^2)^2}$$

$$= \frac{x^3+xy^2-2xy^2}{(x^2+y^2)^2} = \boxed{\frac{x^3-xy^2}{(x^2+y^2)^2}}$$

Example 7: Suppose $f(x,y,z) = 3xyz^2 + \dfrac{1}{xy} - 2z$. Find all the first partial derivatives.

$$f(x,y,z) = 3xyz^2 + (xy)^{-1} - 2z$$

$$f_x(x,y,z) = 3yz^2 - 1(xy)^{-2}(y) + 0 = 3yz^2 - \frac{y}{x^2y^2} = \boxed{3yz^2 - \frac{1}{x^2y}}$$

$$f_y(x,y,z) = 3xz^2 - 1(xy)^{-2}(x) - 0 = 3xz^2 - \frac{x}{x^2y^2} = \boxed{3xz^2 - \frac{1}{xy^2}}$$

$$f_z(x,y,z) = 3xy(2z) + 0 - 2 = \boxed{6xyz - 2}$$

Example 8: Suppose $z = x^2e^{xy^2}$. Find all the first partial derivatives.

$$z = x^2e^{xy^2}$$

$$\frac{\partial z}{\partial x} = x^2\frac{\partial}{\partial x}(e^{xy^2}) + e^{xy^2}\frac{\partial}{\partial x}(x^2)$$

$$= x^2e^{xy^2}y^2 + e^{xy^2}(2x)$$

$$= \boxed{x^2y^2e^{xy^2} + 2xe^{xy^2}}$$

$$\frac{\partial z}{\partial y} = x^2e^{xy^2}(x\cdot 2y) = \boxed{2x^3ye^{xy^2}}$$

Example 9: Suppose $f(x,y,z) = x^2y^3 + 2xyz - 3yz$. Find all the first partial derivatives at the point $(-2,1,2)$.

$$f_x(x,y,z) = 2xy^3 + 2yz$$

$$f_y(x,y,z) = x^2(3y^2) + 2xz - 3z$$
$$= 3x^2y^2 + 2xz - 3z$$

$$f_z(x,y,z) = 2xy - 3y$$

$$f_x(-2,1,2) = 2(-2)(1)^3 + 2(1)(2)$$
$$= -4+4 = \boxed{0}$$

$$f_y(-2,1,2) = 3(-2)^2(1)^2 + 2(-2)(2) - 3(2)$$
$$= 12 - 8 - 6 = 4 - 6 = \boxed{-2}$$

$$f_z(-2,1,2) = 2(-2)(1) - 3(1)$$
$$= -4 - 3 = \boxed{-7}$$

Example 10: Find the slope in the x- and y-directions of the surface given by $f(x,y) = x\sin(x+y)$ at the point $\left(\dfrac{\pi}{2}, \dfrac{\pi}{3}\right)$.

$$f_x(x,y) = x\frac{\partial}{\partial x}(\sin(x+y)) + \sin(x+y)\frac{\partial}{\partial x}(x)$$

$$= x[\cos(x+y)](1) + \sin(x+y)(1)$$

$$= x\cos(x+y) + \sin(x+y)$$

$$f_y(x,y) = x\cos(x+y)(1)$$
$$= x\cos(x+y)$$

Slope in x-direction:

$$f_x\left(\frac{\pi}{2},\frac{\pi}{3}\right) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}+\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{2}+\frac{\pi}{3}\right)$$

$$= \frac{\pi}{2}\cos\left(\frac{3\pi}{6}+\frac{2\pi}{6}\right) + \sin\left(\frac{3\pi}{6}+\frac{2\pi}{6}\right)$$

$$= \frac{\pi}{2}\cos\left(\frac{5\pi}{6}\right) + \sin\left(\frac{5\pi}{6}\right)$$

$$= \frac{\pi}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{2} = -\frac{\sqrt{3}\pi}{4} + \frac{2}{4} = \boxed{\frac{-\sqrt{3}\pi + 2}{4}}$$

$$f_y\left(\frac{\pi}{2},\frac{\pi}{3}\right) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}+\frac{\pi}{3}\right)$$

$$= \frac{\pi}{2}\cos\left(\frac{5\pi}{6}\right)$$

$$= \frac{\pi}{2}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{-\sqrt{3}\pi}{4}}$$

Higher-order partial derivatives:

w.r.t. = with respect to

The second partial derivatives of $z = f(x, y)$ are defined as follows:

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$ Differentiate 1st w.r.t. x, then w.r.t. x

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$ Diff. 1st w.r.t. y, then w.r.t. y

Mixed 2nd Partials

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$ Diff. 1st w.r.t. x, then w.r.t. y

$$\cancel{f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}}$$ ~~Diff. 1st~~ w.r.t.

$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$ Diff. 1st w.r.t. y, then w.r.t. x

Example 11: Suppose $f(x, y) = x\sin(4x - 3y)$. Find all the second partial derivatives.

$$f_x(x,y) = x\frac{\partial}{\partial x}(\sin(4x-3y)) + \sin(4x-3y)\frac{\partial}{\partial x}(x)$$

$$= x(\cos(4x-3y))(4) + \sin(4x-3y)(1)$$

$$= 4x\cos(4x-3y) + \sin(4x-3y)$$

$$f_y(x,y) = x(\cos(4x-3y))(-3) = -3x\cos(4x-3y)$$

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left[4x\cos(4x-3y) + \sin(4x-3y)\right]$$

$$= 4x\frac{\partial}{\partial x}[\cos(4x-3y)] + \cos(4x-3y)\frac{\partial}{\partial x}(4x) + \cos(4x-3y)(4)$$

$$= 4x(-\sin(4x-3y))(4) + \cos(4x-3y)(4) + 4\cos(4x-3y)$$

$$= \boxed{-16x\sin(4x-3y) + 8\cos(4x-3y)}$$

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Example 12: Suppose $f(x, y, z) = \dfrac{3z^2}{x + 2y}$. Find all the second partial derivatives.

$$f(x,y,z) = 3z^2(x+2y)^{-1}$$

$$f_x(x,y,z) = 3z^2(-1)(x+2y)^{-2}(1) = -3z^2(x+2y)^{-2}$$

$$f_y(x,y,z) = 3z^2(-1)(x+2y)^{-2}(2) = -6z^2(x+2y)^{-2}$$

$$f_z(x,y,z) = 6z(x+2y)^{-1}$$

$$f_{xz}(x,y,z) = \frac{\partial}{\partial z}\left(-3z^2(x+2y)^{-2}\right) = \boxed{-6z(x+2y)^{-2}}$$

ok

$$f_{zx}(x,y,z) = \frac{\partial}{\partial x}\left(6z(x+2y)^{-1}\right) = 6z(-1)(x+2y)^{-2}(1) = \boxed{-6z(x+2y)^{-2}}$$

Ex 11 cont'd: $f(x,y) = x\sin(4x-3y)$

$f_x(x,y) = 4x\cos(4x-3y) + \sin(4x-3y)$

$f_y(x,y) = x(\cos(4x-3y))(-3) = -3x\cos(4x-3y)$

$f_{yy}(x,y) = \frac{\partial}{\partial y}\left[-3x\cos(4x-3y)\right]$

$= -3x(-\sin(4x-3y))(-3)$

$= \boxed{-9x\sin(4x-3y)}$

$f_{xy}(x,y) = \frac{\partial}{\partial y}\left[4x\cos(4x-3y) + \sin(4x-3y)\right]$

$= 4x(-\sin(4x-3y))(-3) + \cos(4x-3y)(-3)$

$= \boxed{12x\sin(4x-3y) - 3\cos(4x-3y)}$

✓ they match!

$f_{yx}(x,y) = \frac{\partial}{\partial x}(-3x\cos(4x-3y))$

$= -3x\frac{\partial}{\partial x}(\cos(4x-3y)) + \cos(4x-3y)\frac{\partial}{\partial x}(-3x)$

$= -3x(-\sin(4x-3y))(4) + (\cos(4x-3y))(-3)$

$= \boxed{12x\sin(4x-3y) - 3\cos(4x-3y)}$

Theorem: Equality of Mixed Partial Derivatives
(sometimes known as Clairut's Theorem).

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then

$$f_{xy}(x,y) = f_{yx}(x,y) \text{ for every } (x,y) \text{ in } R.$$

This theorem also applies to third- and higher order derivatives, and to functions of three or more variables. As long as all the higher-order partial derivatives are continuous, all the mixed partial derivatives of that order will be equal.

$w(x,y,z) = x^3 + y^4 z + x^2 y^3 z^2 - z^4$

Example 13: Suppose $w(x,y,z) = \cancel{e^{x^2yz}}$. Find $w_{xyz}(x,y,z)$, $w_{zyx}(x,y,z)$, $w_{xyx}(x,y,z)$, and $w_{xxy}(x,y,z)$.

$w_x(x,y,z) = 3x^2 + 0 + 2xy^3z^2 + 0 = 3x^2 + 2xy^3z^2$

$w_z(x,y,z) = 0 + y^4 + x^2y^3(2z) - 4z^3 = y^4 + 2x^2y^3z - 4z^3$

$w_{xy}(x,y,z) = \frac{\partial}{\partial y}(3x^2 + 2xy^3z^2) = 0 + 2xz^2(3y^2) = 6xy^2z^2$

$w_{zy}(x,y,z) = \frac{\partial}{\partial y}(y^4 + 2x^2y^3z - 4z^3) = 4y^3 + 2x^2z(3y^2) - 0$
$= 4y^3 + 6x^2y^2z$

$w_{xx}(x,y,z) = \frac{\partial}{\partial x}(3x^2 + 2xy^3z^2) = 6x + 2y^3z^2$

$w_{xyz}(x,y,z) = \frac{\partial}{\partial z}(6xy^2z^2) = 6xy^2(2z) = \boxed{12xy^2z}$ = ✓

$w_{zyx}(x,y,z) = \frac{\partial}{\partial x}(4y^3 + 6x^2y^2z) = 0 + 12xy^2z = \boxed{12xy^2z}$

$w_{xyx} = \frac{\partial}{\partial x}(6xy^2z^2)$
$= \boxed{6y^2z^2}$ ✓ =

$w_{xxy} = \frac{\partial}{\partial y}(6x + 2y^3z^2)$
$= 0 + 6y^2z^2$
$= \boxed{6y^2z^2}$

Example 14: Determine whether the following functions satisfy the partial differential equation (PDE) $u_{xx} + u_{yy} = 0$, known as Laplace's equation.

a) $u = x^3 + 3xy^2$
b) $u = e^{-x}\cos y - e^{-y}\cos x$

(a) $u_x = 3x^2 + 3y^2$
$u_y = 0 + 3x(2y) = 6xy$
$u_{xx} = 6x$
$u_{yy} = 6x$
$u_{xx} + u_{yy} = 6x + 6x = 12x \neq 0$
$\boxed{\text{NO}}$

(b) $u_x = e^{-x}(-1)\cos y - e^{-y}(-\sin x)$
$= -e^{-x}\cos y + e^{-y}\sin x$
$u_y = e^{-x}(-\sin y) - e^{-y}(-1)\cos x$
$= -e^{-x}\sin y + e^{-y}\cos x$
$u_{xx} = -e^{-x}(-1)\cos y + e^{-y}\cos x$
$= e^{-x}\cos y + e^{-y}\cos x$
$u_{yy} = -e^{-x}\cos y + e^{-y}(-1)\cos x$
$= -e^{-x}\cos y - e^{-y}\cos x$

these add up to 0, so $\boxed{\text{yes}}$