13.4: Differentials

Essential question in calculus: If we make an incremental change in one variable, what happens to the other variables?

Increments: Δx, Δy, Δz, etc.

Differentials: dx, dy, dz, etc.

For $y = f(x)$, $\Delta y = f(x+\Delta x) - f(x)$
For $z = f(x,y)$, $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$.

For the independent variable(s), we define the differential to be equal to the increment. For the dependent variable, we define the differential in terms of the derivative.

Calc I For $y = f(x)$, we let $dx = \Delta x$. Then the differential of y is $dy = f'(x)\,dx$.

$\frac{dy}{dx} = f'(x)$

The differential dy has two uses:
- (a) as an approximation for the increment Δy. ($dy \approx \Delta y$)
- (b) as a building block for calculus concepts (integration, etc).

To extend the concept of differentials to functions of two or more variables, we need to use partial derivatives.

<u>Definition</u>:

If $z = f(x,y)$, then the differentials of the independent variables x and y are the increments:

$$dx = \Delta x \qquad\qquad dy = \Delta y$$

The <u>total differential</u> of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

<u>Note</u>: For small increments of z, $dz \approx \Delta z$ (for differentiable functions).

Example 1: Find the total differential for $z = x^2 y + x^4 - 5y^3$.

$\frac{\partial z}{\partial x} = 2xy + 4x^3$

$\frac{\partial z}{\partial y} = x^2 + 0 - 15y^2$

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$\boxed{dz = (2xy + 4x^3)dx + (x^2 - 15y^2)dy}$

$$\frac{\partial w}{\partial x} = 2xyz^2 + 0$$

$$\frac{\partial w}{\partial y} = x^2z^2 + (\cos(yz))(z) = x^2z^2 + z\cos(yz)$$

Example 2: Find the total differential for $w = x^2yz^2 + \sin(yz)$.

$$\frac{\partial w}{\partial z} = x^2y(2z) + (\cos(yz))(y) = 2x^2yz + y\cos(yz)$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

$$= \boxed{2xyz^2\,dx + (x^2z^2 + z\cos(yz))dy + (2x^2yz + y\cos(yz))dz}$$

Differentiability:

Definition: A function *f* given by $z = f(x, y)$ is *differentiable at* (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

where both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

The function *f* is differentiable in a region *R* if it is differentiable at each point in *R*.

Important Note: Existence of the partial derivatives f_x and f_y at the point (x_0, y_0) does NOT guarantee the function is differentiable at (x_0, y_0).

However, if the partial derivatives are continuous in a region, then the function is differentiable in the region:

Theorem: If *f* is a function of *x* and *y*, and if f_x and f_y are continuous on an open region *R*, then f is differentiable on *R*.

Using the total differential as an approximation for the increment:

Example 3: Suppose $z = f(x, y) = \dfrac{y}{x}$. Calculate the increment Δz as *y* changes from 2 to 2.1, and *x* changes from 1 to 1.03. Use the total differential dz as an approximation for Δz, and compare it to the actual value of Δz.

$$z = \frac{y}{x} = yx^{-1}$$

$$dy = \Delta y = 0.1$$

$$dx = \Delta x = 0.03$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= y(-1x^{-2})\,dx + x^{-1}\,dy$$

$$= -\frac{y}{x^2}\,dx + \frac{1}{x}\,dy$$

$$dz\Big|_{\substack{(x,y)=(1,2)\\ dx=0.03\\ dy=0.1}} = -\frac{2}{1^2}(0.03) + \frac{1}{1}(0.1)$$

$$= -0.06 + 0.1$$

$$= \boxed{0.04}$$

Error analysis:

When using a formula to compute a physical quantity, any measurement error in the independent variables will cause propogated error in the dependent variable. We can use the total differential to estimate the propogated error.

> Estimating propagated error:
>
> Suppose that x and y are the measured values of two variables, and that $dx = \Delta x$ and $dy = \Delta y$ are the maximum possible measurement errors. Then the estimated propogated error in $z = f(x, y)$ is
>
> Estimated propagated error: $dz = f_x(x, y)\, dx + f_y(x, y)\, dy$
>
> Estimated relative error: $\dfrac{dz}{z}$

Example 4: The measurements of the height and inside radius of a right circular cone are 50 feet and 30 feet, respectively. The maximum possible error in each measurement is about 3 inches per each 10 feet of measured length. Approximate the maximum propagated error and the relative error in computing the volume of the cone.

$$V(r, h) = \frac{1}{3}\pi r^2 h$$

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$= \frac{1}{3}\pi h(2r)\, dr + \frac{1}{3}\pi r^2\, dh$$

$$= \frac{2}{3}\pi r h\, dr + \frac{1}{3}\pi r^2 dh$$

$$dV\Big|_{\substack{r=30' \\ h=50'}} = \frac{2}{3}\pi(30\text{ ft})(50\text{ ft})(0.75\text{ ft}) + \frac{1}{3}\pi(30\text{ ft})^2(1.25\text{ ft})$$

$dr = .75'$
$dh = 1.25'$

$$= 750\pi\text{ ft}^3 + 375\pi\text{ ft}^3 = \boxed{1125\pi\text{ ft}^3}$$

Max propogated error

$$\text{Relative Error} = \frac{dV}{V} = \frac{1125\pi\text{ ft}^3}{\frac{1}{3}\pi(30\text{ ft})^2(50\text{ ft})}$$

$$= 0.075 \Rightarrow \boxed{7.5\%}$$

Find dr and dh:

Radius:

$r = 30$ ft $\Rightarrow dr = \dfrac{3\text{ in}}{10'\text{ segment}}$ (3 10' segments)

$= 9\,\cancel{\text{in}}\left(\dfrac{1\text{ ft}}{12\,\cancel{\text{in}}}\right)$

$= 0.75$ ft

height:

$h = 50$ ft $\Rightarrow$ 5 10' segments

$\Rightarrow dh = 5(3\text{ in}) = 15\,\cancel{\text{in}}\left(\dfrac{1\text{ ft}}{12\,\cancel{\text{in}}}\right)$

$= 1.25$ ft