13.5: Chain Rules for Functions of Several Variables

Chain Rule (One Independent Variable):

Suppose $w = f(x, y)$, where f is a differentiable function of x and y. Also suppose $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t. Then, w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Example 1: Suppose that $w = \cos(x - y)$, $x = t^2$, $y = 1$. Find $\dfrac{dw}{dt}$.

Example 2: Suppose that $w = xy\cos(z)$, $x = t$, $y = t^2$, $z = \arccos(t)$. Find $\dfrac{dw}{dt}$.

<u>Chain Rule (Two Independent Variables)</u>:

Suppose $w = f(x, y)$, where f is a differentiable function of x and y. Also suppose $x = g(s,t)$ and $y = h(s,t)$, such that the first partial derivatives $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, and $\frac{\partial y}{\partial t}$ all exist.

Then, $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ both exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

<u>Example 3:</u> Suppose that $w = \sin(2x + 3y)$, $x = s + t$, $y = s - t$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$ at the point where $s = 0$, $t = \frac{\pi}{4}$.

<u>Example 4:</u> Suppose that $w = \sqrt{25 - 5x^2 - 5y^2}$, $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.

<u>Example 5:</u> Suppose that $w = x^2 - 2xy + y^2$, $x = r + \theta$, $y = r - \theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.

Chain Rule (Implicit Differentiation):

Suppose the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x.
Then,

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \text{ provided } F_y(x, y) \neq 0.$$

Suppose the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y.

Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \text{ provided } F_z(x, y, z) \neq 0.$$

Example 6: Suppose that $2x^2 - 3y^3 = xy^2 - 5$. Find $\frac{dy}{dx}$.

Example 7: Suppose that $\ln(x^2 + y^2) + 2xy = 8$. Find $\frac{dy}{dx}$.

Example 8: Suppose that $z^3 + 2xy^2 = \ln z + x^3y^4z^2$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Example 9: Suppose that $z = e^x \sin(y+z)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Example 10: Suppose that $yz^4 + x^2y^3 = e^{xyz}$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Example 11: Suppose that $x \ln y + y^2 z + z^2 = 8$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.