13.5: Chain Rules for Functions of Several Variables

Chain Rule (One Independent Variable):

Suppose $w = f(x, y)$, where f is a differentiable function of x and y. Also suppose $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t. Then, w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Note: multiplying by dt results in:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$$

(the total differential)

Example 1: Suppose that $w = \cos(x - y)$, $x = t^2$, $y = 1$. Find $\frac{dw}{dt}$.

Using Chain Rule:

$$\frac{\partial w}{\partial x} = -\sin(x-y)(1) = -\sin(x-y)$$

$$\frac{\partial w}{\partial y} = -\sin(x-y)(-1) = \sin(x-y)$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

$$= (-\sin(x-y))(2t) + (\sin(x-y))(0)$$

$$= -2t\sin(x-y)$$

$$= \boxed{-2t\sin(t^2-1)}$$ [substituting $x=t^2, y=1$]

OR

$$w = \cos(x-y)$$

$$= \cos(t^2-1)$$

$$\frac{dw}{dt} = -\sin(t^2-1)(2t)$$

$$= \boxed{-2t\sin(t^2-1)}$$

✓ OK

Example 2: Suppose that $w = xy\cos(z)$, $x = t$, $y = t^2$, $z = \arccos(t)$. Find $\frac{dw}{dt}$.

Chain Rule:

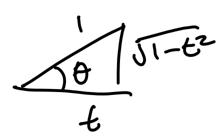
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= (y\cos z)(1) + (x\cos z)(2t) + (-xy\sin z)\left(-\frac{1}{\sqrt{1-t^2}}\right)$$

$$= y\cos z + 2tx\cos z + \frac{xy\sin z}{\sqrt{1-t^2}}$$

$$= t^2\cos(\arccos t) + 2t(t)\cos(\arccos t) + \frac{t(t^2)\sin(\arccos t)}{\sqrt{1-t^2}}$$

$$= t^2(t) + 2t^2(t) + \frac{t^3\sqrt{1-t^2}}{\sqrt{1-t^2}} = t^3 + 2t^3 + t^3 = \boxed{4t^3}$$



$\theta = \arccos t$

$\sin\theta = \frac{\sqrt{1-t^2}}{1}$

Substituting for t in w:

$$w = xy\cos z = t(t^2)\cos(\arccos t) = t^3(t) = t^4$$

$$\frac{dw}{dt} = \boxed{4t^3}$$

Ex $2\frac{1}{2}$) $w = xy\cos z$, $x = t$, $y = \sqrt{t^2+1}$, $z = t^2$

$w(t) = t\sqrt{t^2+1}\ \cos(t^2)$

3-factor product rule problem.
Would be a pain!

Chain Rule:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= (y\cos z)(1) + (x\cos z)\left(\frac{1}{2}\right)(t^2+1)^{-1/2}(2t) - (xy\sin z)(2t)$$

Then substitute for x, y and z
to get t.

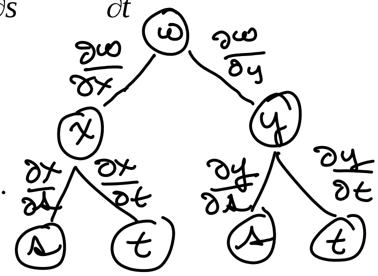
For this problem,
Chain rule is easiest method.

<u>Chain Rule (Two Independent Variables)</u>:

Suppose $w = f(x, y)$, where f is a differentiable function of x and y. Also suppose $x = g(s,t)$ and $y = h(s,t)$, such that the first partial derivatives $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, and $\frac{\partial y}{\partial t}$ all exist.

Then, $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ both exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$



Note: $2x+3y = 2(s+t)+3(s-t) = 2s+2t+3s-3t = 5s-t$

Example 3: Suppose that $w = \sin(2x+3y)$, $x = s+t$, $y = s-t$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$ at the point where $s = 0$, $t = \frac{\pi}{4}$.

$w_x = \cos(2x+3y)(2) = 2\cos(2x+3y)$

$w_y = \cos(2x+3y)(3) = 3\cos(2x+3y)$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} = (2\cos(2x+3y))(1) + (3\cos(2x+3y))(-1)$$
$$= 2\cos(2x+3y) - 3\cos(2x+3y)$$
$$= 2\cos(5s-t) - 3\cos(5s-t) = \boxed{-\cos(5s-t)}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} = (2\cos(2x+3y))(1) + (3\cos(2x+3y))(1)$$
$$= 5\cos(2x+3y) = \boxed{5\cos(5s-t)}$$

$s=0,\ t=\frac{\pi}{4} \Rightarrow \frac{\partial w}{\partial t} = -\cos\left(0-\frac{\pi}{4}\right) = \boxed{-\sqrt{2}/2}$

$\left.\frac{\partial w}{\partial s}\right|_{s=0,\ t=\pi/4} = 5\cos\left(-\frac{\pi}{4}\right) = \boxed{\frac{5\sqrt{2}}{2}}$

Example 4: Suppose that $w = \sqrt{25-5x^2-5y^2}$, $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.

$w = (25-5x^2-5y^2)^{1/2}$

$w_x = \left(\frac{1}{2}\right)(25-5x^2-5y^2)^{-1/2}(-10x)$

$= -5x(25-5x^2-5y^2)^{-1/2}$

$w_y = \frac{1}{2}(25-5x^2-5y^2)^{-1/2}(-10y)$

$= -5y(25-5x^2-5y^2)^{-1/2}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$
$$= -5x(25-5x^2-5y^2)^{-1/2}\cos\theta - 5y(25-5x^2-5y^2)^{-1/2}\sin\theta$$

Note: $25-5x^2-5y^2 = 25-5r^2\cos^2\theta - 5r^2\sin^2\theta$
$= 25-5r^2$

Would need to put r, θ back in.

Example 5: Suppose that $w = x^2-2xy+y^2$, $x = r+\theta$, $y = r-\theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.

(we did not do this one in class)

$\frac{\partial w}{\partial x} = 2x-2y$, $\frac{\partial w}{\partial y} = -2x+2y$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$
$$= (2x-2y)(1) + (-2x+2y)(1)$$
$$= 2x-2y-2x+2y = \boxed{0}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$
$$= (2x-2y)(1) + (-2x+2y)(-1)$$
$$= 2x-2y+2x-2y = 4x-4y$$
$$= 4(r+\theta) - 4(r-\theta) = 4r+4\theta-4r+4\theta$$
$$= \boxed{8\theta}$$

<u>Chain Rule (Implicit Differentiation)</u>:

Suppose the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x.
Then,

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}, \text{ provided } F_y(x,y) \neq 0.$$

Suppose the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y.

Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}, \text{ provided } F_z(x,y,z) \neq 0.$$

See next page

Example 6: Suppose that $2x^2 - 3y^3 = xy^2 - 5$. Find $\frac{dy}{dx}$.

Get 1 side 0: $2x^2 - 3y^3 - xy^2 + 5 = 0$

$F(x,y) = 2x^2 - 3y^3 - xy^2 + 5$

$F_x(x,y) = 4x - y^2$

$F_y(x,y) = -9y^2 - x(2y)$

$= -9y^2 - 2xy$

$$\frac{dy}{dx} = -\frac{F_x(x,y,z)}{F_y(x,y,z)} = -\frac{4x - y^2}{-9y^2 - 2xy} = \frac{-(4x-y^2)}{-(9y^2+2xy)} = \boxed{\frac{4x - y^2}{9y^2 + 2xy}}$$

Example 7: Suppose that $\ln(x^2 + y^2) + 2xy = 8$. Find $\frac{dy}{dx}$.

$F(x,y) = \ln(x^2+y^2) + 2xy - 8 = 0$

$F_x(x,y) = \frac{1}{x^2+y^2}(2x) + 2y = \frac{2x}{x^2+y^2} + 2y$

$F_y(x,y) = \frac{1}{x^2+y^2}(2y) + 2x = \frac{2y}{x^2+y^2} + 2x$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{\frac{2x}{x^2+y^2} + 2y}{\frac{2y}{x^2+y^2} + 2x}$$

$$= -\frac{\frac{2x}{x^2+y^2} + 2y}{\frac{2y}{x^2+y^2} + 2x}\left(\frac{x^2+y^2}{x^2+y^2}\right) = -\frac{2x + 2y(x^2+y^2)}{2y + 2x(x^2+y^2)}$$

$$= -\frac{2x + 2x^2y + 2y^3}{2y + 2x^3 + 2y^2x}$$

$$= \boxed{-\frac{x + x^2y + y^3}{y + x^3 + xy^2}}$$

Example 8: Suppose that $z^3 + 2xy^2 = \ln z + x^3y^4z^2$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$F(x,y,z) = z^3 + 2xy^2 - \ln z - x^3y^4z^2$

$F_x(x,y,z) = 2y^2 - 3x^2y^4z^2$

$F_y(x,y,z) = 4xy - 4x^3y^3z^2$

$F_z(x,y,z) = 3z^2 - \frac{1}{z} - 2x^3y^4z$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = \boxed{-\frac{2y^2 - 3x^2y^4z^2}{3z^2 - \frac{1}{z} - 2x^3y^4z}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = \boxed{-\frac{4xy - 4x^3y^3z^3}{3z^2 - \frac{1}{z} - 2x^3y^4z}}$$

Why is $\dfrac{dy}{dx} = -\dfrac{F_x(x,y)}{F_y(x,y)}$?

Suppose $w = F(x,y) = 0$.

$$\frac{\partial}{\partial x}\left(F(x,y)\right) = \frac{\partial}{\partial x}(0)$$

$$\frac{\partial}{\partial x}(w) = \frac{\partial}{\partial x}(0)$$

$$\frac{\partial w}{\partial x}\frac{dx}{dx} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\frac{\partial w}{\partial x}(1) + \frac{\partial w}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$F_x(x,y) + F_y(x,y)\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$: $\quad F_y(x,y)\dfrac{dy}{dx} = -F_x(x,y)$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$

Examples 9 – 11 were not worked in class... I did them later.

Example 9: Suppose that $z = e^x \sin(y+z)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$F(x,y,z) = e^x \sin(y+z) - z = 0$

$F_x(x,y,z) = e^x \sin(y+z)$

$F_y(x,y,z) = e^x \cos(y+z)$

$F_z(x,y,z) = e^x \cos(y+z) - 1$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = \boxed{-\frac{e^x \sin(y+z)}{e^x \cos(y+z) - 1}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = \boxed{-\frac{e^x \cos(y+z)}{e^x \cos(y+z) - 1}}$$

Example 10: Suppose that $yz^4 + x^2y^3 = e^{xyz}$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$F(x,y,z) = yz^4 + x^2y^3 - e^{xyz}$

$F_x(x,y,z) = 2xy^3 - e^{xyz}\, yz$

$F_y(x,y,z) = z^4 + 3x^2y^2 - e^{xyz}\, xz$

$F_z(x,y,z) = 4yz^3 - e^{xyz}\, xy$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = \boxed{-\frac{2xy^3 - yze^{xyz}}{4yz^3 - xye^{xyz}}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = \boxed{-\frac{z^4 + 3x^2y^2 - xze^{xyz}}{4yz^3 - xye^{xyz}}}$$

Example 11: Suppose that $x \ln y + y^2 z + z^2 = 8$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$F(x,y,z) = x \ln y + y^2 z + z^2 - 8 = 0.$

$F_x(x,y,z) = \ln y$

$F_y(x,y,z) = \frac{x}{y} + 2yz$

$F_z = y^2 + 2z$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = \boxed{-\frac{\ln y}{y^2 + 2z}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = -\frac{\frac{x}{y} + 2yz}{y^2 + 2z} = -\frac{\frac{x}{y} + 2yz}{y^2 + 2z}\left(\frac{y}{y}\right) = \boxed{-\frac{x + 2y^2z}{y^3 + 2zy}}$$