13.6: Directional Derivatives and Gradients

Given $z = f(x, y)$, the partial derivatives f_x and f_y represent the rates of change of *z* in the *x*- and *y*-directions. That is, these are the rates of change of *z* along the vectors **i** and **j**.

We can calculate the rate of change in *z* along any vector **u**, called the *directional derivative* in the direction of **u**.

Definition: Directional Derivative

Let *f* be a function of two variables *x* and *y*, and let $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ be a unit vector. Then, the directional derivative of *f* in the direction of **u** is

$$D_{\mathbf{u}} f(x, y) = \lim_{t \to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t},$$

provided this limit exists.

The above limit definition is not very practical for calculating directional derivatives. Instead, we'll use this theorem:

Theorem:

Let $f(x, y)$ be a differentiable function.

Then, the directional derivative of *f* in the direction of $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ is

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \\ &= \langle f_x, f_y \rangle \cdot \langle \cos\theta, \sin\theta \rangle. \end{aligned}$$

Example 1: Suppose $f(x, y) = x^2 - 2xy^2 + 3y^4$. Find the directional derivative in the direction $\theta = \frac{\pi}{3}$ at the point $(1, 2)$.

Example 2: Find the directional derivative for $f(x,y)=xy$ in the direction $\mathbf{u}=\left\langle \frac{3}{5},\frac{2}{5}\right\rangle$ at the point $(2,3)$.

Example 3: Find the directional derivative for $f(x,y)=e^x \sin y$ in the direction $\mathbf{v}=\langle -1,5\rangle$ at the point $\left(1,\frac{\pi}{2}\right)$.

Example 4: Find the directional derivative for $f(x,y)=x^4+5y^2$ at point $P(2,1)$ in the direction toward the point $Q(-4,3)$.

The gradient vector:

<u>Definition</u>: <u>The Gradient</u>

Let $z = f(x, y)$ be a function of *x* and *y* such that f_x and f_y exist. Then the gradient of *f*, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = \left\langle f_x(x, y), f_y(x, y) \right\rangle.$$

<u>Example 5:</u> Find the gradient of $z = \ln(x^2 - y)$ at the point $(2,3)$.

<u>Theorem</u>: Dot Product Form of the Directional Derivative

Let $f(x, y)$ be a differentiable function. Then, the directional derivative of *f* in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

<u>Example 6:</u> Find the directional derivative for $f(x, y) = 3x^2 - y^2 + 4$ at point $P(-1, 4)$ in the direction toward the point $Q(3, 6)$.

Directional Derivative and Gradient for Functions of Three Variables:

Let f be a function of three variables with continuous first partial derivatives.

Then the gradient of f is the vector

$$\nabla f(x,y) = \left\langle f_x(x,y,z),\, f_y(x,y,z),\, f_z(x,y,z) \right\rangle.$$

The directional derivative of f in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}\,.$$

Example 7: Find the directional derivative for $f(x,y,z) = xy + yz + xz$ at the point $P(1,2,-1)$ in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Example 8: Find the directional derivative for $f(x,y,z) = e^{2x}\cos y \sin z$ at the point $P\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$ in the direction of $\mathbf{v} = \langle 2, -1, 2 \rangle$.

Finding the directions of maximum and minimum increase:

<u>Theorem</u>: <u>Properties of the Gradient</u>

Suppose the function f is differentiable at the point (x, y). Then,

1) If $\nabla f(x,y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x,y) = 0$ for all $\mathbf{u}$.

2) The direction of <u>maximum increase</u> of f is given by $\nabla f(x,y)$.
   The maximum value of $D_{\mathbf{u}} f(x,y)$ is $\|\nabla f(x,y)\|$.

3) The direction of <u>minimum increase</u> of f is given by $-\nabla f(x,y)$.
   The minimum value of $D_{\mathbf{u}} f(x,y)$ is $-\|\nabla f(x,y)\|$.

<u>Note</u>: These properties also hold for the gradient of a three-variable function $f(x, y, z)$.

<u>Example 9:</u> Find the maximum value of the directional derivative for $f(x,y) = x\tan y$ at the point $P\left(2, \dfrac{\pi}{4}\right)$. What is the direction of maximum increase?

<u>Example 10:</u> Find the maximum and minimum rate of increase in the function $f(x,y,z) = xy + yz + xz$ at the point $P(1,2,-1)$.

Finding a vector normal to the level curves of a function:

Theorem:

Let f be a function differentiable at (x_0, y_0) such that $\nabla f(x_0, y_0) \neq \mathbf{0}$. Then, $\nabla f(x_0, y_0)$ is orthogonal to the level curve that passes through the point (x_0, y_0).

Example 11: For the function $f(x, y) = x + y^2$, f ind a normal vector to the level curve through the point $(1,3)$.