13.6: Directional Derivatives and Gradients

Given $z = f(x, y)$, the partial derivatives f_x and f_y represent the rates of change of *z* in the *x*- and *y*-directions. That is, these are the rates of change of *z* along the vectors **i** and **j**.

We can calculate the rate of change in *z* along any vector **u**, called the *directional derivative* in the direction of **u**.

> Definition: Directional Derivative
>
> Let *f* be a function of two variables *x* and *y*, and let $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ be a unit vector. Then, the directional derivative of *f* in the direction of **u** is
>
> $$D_{\mathbf{u}} f(x, y) = \lim_{t \to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t},$$
>
> provided this limit exists.

The above limit definition is not very practical for calculating directional derivatives. Instead, we'll use this theorem:

> Theorem:
>
> Let $f(x, y)$ be a differentiable function.
>
> Then, the directional derivative of *f* in the direction of $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ is
>
> $$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \\ &= \langle f_x, f_y \rangle \cdot \langle \cos\theta, \sin\theta \rangle. \end{aligned}$$

Example 1: Suppose $f(x, y) = x^2 - 2xy^2 + 3y^4$. Find the directional derivative in the direction $\theta = \frac{\pi}{3}$ at the point $(1, 2)$. $f_x(x,y) = 2x - 2y^2$

$f_y(x,y) = -4xy + 12y^3$

At the point $(1,2)$: $f_x(1,2) = 2(1) - 2(2)^2 = 2 - 8 = -6$

$f_y(1,2) = -4(1)(2) + 12(2)^3 = -8 + 96 = 88$

$D_{\vec{u}} f(x,y) = \langle f_x(1,2), f_y(1,2) \rangle \cdot \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle$

$= \langle -6, 88 \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = -6\left(\frac{1}{2}\right) + 88\left(\frac{\sqrt{3}}{2}\right)$

$= \boxed{-3 + 44\sqrt{3}}$

$\|\vec{u}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{4}{25}} = \sqrt{\frac{13}{25}} = \frac{\sqrt{13}}{5}$

Example 2: Find the directional derivative for $f(x, y) = xy$ in the direction $\mathbf{u} = \left\langle \frac{3}{5}, \frac{2}{5} \right\rangle$ at the point $(2,3)$.

$\vec{u} = \langle \frac{3}{5}, \frac{2}{5} \rangle$

Get a unit vector: $\vec{v} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle \frac{3}{5}, \frac{2}{5} \rangle}{\frac{\sqrt{13}}{5}} = \frac{5}{\sqrt{13}} \langle \frac{3}{5}, \frac{2}{5} \rangle$

$= \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle$ (unit vector)

$= \langle \cos\theta, \sin\theta \rangle$

$f_x(x,y) = y$, $f_y(x,y) = x$; $f_x(2,3) = 3$, $f_y(2,3) = 2$

$D_{\vec{v}} f(x,y) = \langle f_x, f_y \rangle \cdot \langle \cos\theta, \sin\theta \rangle$

$= \langle 3, 2 \rangle \cdot \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle = \frac{9}{\sqrt{13}} + \frac{4}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \boxed{\sqrt{13}}$

Example 3: Find the directional derivative for $f(x, y) = e^x \sin y$ in the direction $\mathbf{v} = \langle -1, 5 \rangle$ at the point $\left(1, \frac{\pi}{2}\right)$.

Make a unit vector in the direction of $\vec{v}$:

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle -1, 5 \rangle}{\sqrt{1+25}} = \langle -\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \rangle$

$f(x,y) = e^x \sin y$

$f_x(x,y) = e^x \sin y$

$f_y(x,y) = e^x \cos y$

$f_x(1, \frac{\pi}{2}) = e^1 \sin\frac{\pi}{2} = e$

$f_y(1, \frac{\pi}{2}) = e^1 \cos\frac{\pi}{2} = 0$

$D_{\vec{u}} f(1, \frac{\pi}{2}) = \langle e, 0 \rangle \cdot \langle -\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \rangle$

$= \boxed{-\frac{e}{\sqrt{26}}}$

Example 4: Find the directional derivative for $f(x, y) = x^4 + 5y^2$ at point $P(2,1)$ in the direction toward the point $Q(-4,3)$.

$\vec{PQ} = \langle -4-2, 3-1 \rangle = \langle -6, 2 \rangle$

Find a unit vector:

$\vec{u} = \frac{\vec{PQ}}{\|\vec{PQ}\|} = \frac{\langle -6, 2 \rangle}{\sqrt{36+4}} = \langle -\frac{6}{\sqrt{40}}, \frac{2}{\sqrt{40}} \rangle = \langle -\frac{6}{2\sqrt{10}}, \frac{2}{2\sqrt{10}} \rangle = \langle -\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \rangle$

$f_x(x,y) = 4x^3$

$f_y(x,y) = 10y$

$f_x(2,1) = 4(2)^3 = 32$

$f_y(2,1) = 10(1) = 10$

$D_{\vec{u}} f(2,1) = \langle 32, 10 \rangle \cdot \langle -\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \rangle$

$= -\frac{96}{\sqrt{10}} + \frac{10}{\sqrt{10}} = -\frac{86}{\sqrt{10}}$

$= -\frac{86\sqrt{10}}{10} = \boxed{-\frac{43\sqrt{10}}{5}}$

The gradient vector:

Definition: The Gradient

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the gradient of f, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle.$$

sometimes called "del f"

Example 5: Find the gradient of $z = \ln(x^2 - y)$ at the point $(2,3)$.

$$\nabla z = \left\langle \frac{1}{x^2-y}(2x), \frac{1}{x^2-y}(-1) \right\rangle = \left\langle \frac{2x}{x^2-y}, -\frac{1}{x^2-y} \right\rangle$$

$$\nabla z \Big|_{\substack{x=2 \\ y=3}} = \nabla z \Big|_{(x,y)=(2,3)} = \left\langle \frac{2(2)}{2^2-3}, -\frac{1}{2^2-3} \right\rangle = \left\langle \frac{4}{1}, -\frac{1}{1} \right\rangle$$

$$= \boxed{\langle 4, -1 \rangle}$$

Theorem: Dot Product Form of the Directional Derivative

Let $f(x, y)$ be a differentiable function. Then, the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Example 6: Find the directional derivative for $f(x, y) = 3x^2 - y^2 + 4$ at point $P(-1, 4)$ in the direction toward the point $Q(3, 6)$.

$$\nabla f(x,y) = \langle 6x, -2y \rangle$$

$$\nabla f(-1,4) = \langle 6(-1), -2(4) \rangle = \langle -6, -8 \rangle$$

Find unit vector:

$$\vec{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{\langle 4, 2 \rangle}{\sqrt{16+4}}$$

$$= \left\langle \frac{4}{\sqrt{20}}, \frac{2}{\sqrt{20}} \right\rangle = \left\langle \frac{4}{2\sqrt{5}}, \frac{2}{2\sqrt{5}} \right\rangle$$

$$= \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$D_{\vec{u}} f(-1,4) = \nabla f(-1,4) \cdot \vec{u}$$

$$= \langle -6, -8 \rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$= \frac{-12}{\sqrt{5}} - \frac{8}{\sqrt{5}} = \frac{-20}{\sqrt{5}} = \frac{-20\sqrt{5}}{5} = \boxed{-4\sqrt{5}}$$

Directional Derivative and Gradient for Functions of Three Variables:

Let f be a function of three variables with continuous first partial derivatives.

Then the gradient of f is the vector

$$\nabla f(x,y,z)$$
$$\cancel{\nabla f(x,y)} = \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z) \rangle.$$

The directional derivative of f in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}.$$

Example 7: Find the directional derivative for $f(x,y,z) = xy + yz + xz$ at the point $P(1,2,-1)$ in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

$$\nabla f(x,y,z) = \langle y+z,\ x+z,\ y+x \rangle$$
$$\nabla f(1,2,-1) = \langle 2-1,\ 1-1,\ 2+1 \rangle = \langle 1,0,3 \rangle$$
$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 2,1,-1 \rangle}{\sqrt{4+1+1}} = \left\langle \frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}} \right\rangle$$
$$D_{\vec{u}} f(1,2,-1) = \nabla f(1,2,-1) \cdot \vec{u} = \langle 1,0,3 \rangle \cdot \left\langle \frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{-1}{\sqrt{6}} \right\rangle$$
$$= \frac{2}{\sqrt{6}} + 0 - \frac{3}{\sqrt{6}} = \boxed{-\frac{1}{\sqrt{6}}}$$

Example 8: Find the directional derivative for $f(x,y,z) = e^{2x}\cos y \sin z$ at the point $P\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$ in the direction of $\mathbf{v} = \langle 2,-1,2 \rangle$.

$$f(x,y,z) = e^{2x}\cos y \sin z$$
$$\nabla f(x,y,z) = \langle 2e^{2x}\cos y \sin z,\ -e^{2x}\sin y \sin z,\ e^{2x}\cos y \cos z \rangle$$
$$\nabla f\left(0,\frac{\pi}{2},\frac{\pi}{2}\right) = \left\langle 2e^0 \cos\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2},\ -e^0 \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right),\ e^0\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) \right\rangle$$
$$= \langle 2(0)(1),\ -1(1)(1),\ 1(0)(1) \rangle = \langle 0,-1,0 \rangle$$
$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 2,-1,2 \rangle}{\sqrt{4+1+4}} = \left\langle \frac{2}{3},\ -\frac{1}{3},\ \frac{2}{3} \right\rangle$$
$$D_{\vec{u}} f\left(0,\frac{\pi}{2},\frac{\pi}{2}\right) = \langle 0,-1,0 \rangle \cdot \left\langle \frac{2}{3},\ -\frac{1}{3},\ \frac{2}{3} \right\rangle = 0 + \frac{1}{3} + 0 = \boxed{\frac{1}{3}}$$

Finding the directions of maximum and minimum increase:

Theorem: Properties of the Gradient

Suppose the function f is differentiable at the point (x, y). Then,

1) If $\nabla f(x,y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x,y) = 0$ for all $\mathbf{u}$. $\nabla f(x,y) = \vec{0} \Rightarrow D_{\vec{u}} f(x,y) = \vec{\nabla} f \cdot \vec{u} = \vec{0} \cdot \vec{u} = 0$

2) The direction of maximum increase of f is given by $\nabla f(x,y)$.
   The maximum value of $D_{\mathbf{u}} f(x,y)$ is $\|\nabla f(x,y)\|$.

3) The direction of minimum increase of f is given by $-\nabla f(x,y)$.
   The minimum value of $D_{\mathbf{u}} f(x,y)$ is $-\|\nabla f(x,y)\|$.

Note: These properties also hold for the gradient of a three-variable function $f(x, y, z)$.

Why are #2 and #3 true?

If $\nabla f(x,y) \neq \vec{0}$, then let ϕ be the angle between $\nabla f(x,y)$ and unit vector $\vec{u}$.

$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u} = \|\nabla f(x,y)\| \, \|\vec{u}\| \cos\phi = \|\nabla f(x,y)\| (1) \cos\phi$

$= \|\nabla f(x,y)\| \cos\phi$ [because $\vec{u}$ is a unit vector - has magnitude 1]

This is at its max when $\cos\phi = 1$.

This is at its min when $\cos\phi = -1$. $\cos\phi = 1 \Rightarrow \phi = 0 \Rightarrow \nabla f(x,y)$ is in same direction as $\vec{u}$.

$\cos\phi = -1 \Rightarrow \phi = \pi = 180°$, so $\nabla f(x,y)$ is in opposite direction of $\vec{u}$. ($\vec{u} = -\nabla f(x,y)$)

Example 9: Find the maximum value of the directional derivative for $f(x, y) = x \tan y$ at the point $P\left(2, \frac{\pi}{4}\right)$. What is the direction of maximum increase?

$\nabla f(x,y) = \langle \tan y, x\sec^2 y \rangle$

$\nabla f(2, \frac{\pi}{4}) = \langle \tan\frac{\pi}{4}, 2\sec^2(\frac{\pi}{4}) \rangle = \langle 1, 2(\sqrt{2})^2 \rangle$

$= \langle 1, 2 \cdot 2 \rangle = \langle 1, 4 \rangle$

Maximum value of $D_{\vec{u}} f(x,y)$ is $\|\langle 1,4 \rangle\| = \sqrt{1+16} = \boxed{\sqrt{17}}$ Direction of max increase is $\langle 1, 4 \rangle$

Example 10: Find the maximum and minimum rate of increase in the function $f(x, y, z) = xy + yz + xz$ at the point $P(1, 2, -1)$.

See Example 7: $\nabla f(1,2,-1) = \langle 1, 0, 3 \rangle$

Maximum Rate of increase is $\|\nabla f(1,2,-1)\| = \|\langle 1,0,3 \rangle\| = \sqrt{10}$ in direction $\langle 1, 0, 3 \rangle$

Minimum Rate of Increase is $-\sqrt{10}$ in direction $\langle -1, 0, -3 \rangle$.

Finding a vector normal to the level curves of a function:

<u>Theorem</u>:

Let f be a function differentiable at (x_0, y_0) such that $\nabla f(x_0, y_0) \neq \mathbf{0}$. Then, $\nabla f(x_0, y_0)$ is orthogonal to the level curve that passes through the point (x_0, y_0).

<u>Example 11:</u> For the function $f(x, y) = x + y^2$, f ind a normal vector to the level curve through the point $(1,3)$.

$f(1,3) = 1 + 3^2 = 10$

so the $z = 10$ level curve passes through $(1,3)$.

$\nabla f(x,y) = \langle 1, 2y \rangle$

$\nabla f(1,3) = \langle 1, 2(3) \rangle = \langle 1, 6 \rangle$

$\boxed{\vec{N} = \langle 1, 6 \rangle}$

Draw some level curves for $f(x,y) = x + y^2$

$k = x + y^2$

$x = -y^2 + k$

$k = 0 \Rightarrow x = -y^2$

$k = 1 \Rightarrow x = -y^2 + 1$

$k = 2 \Rightarrow x = -y^2 + 2$

$\vdots$

$k = 10 \Rightarrow x = -y^2 + 10$

x	$y = x^2$
0	0
± 1	1
± 2	4
± 3	9