13.7: Tangent Planes and Normal Lines

Suppose a surface in $\mathbb{R}^3$ is represented by the equation $F(x,y,z)=0$, and that $P(x_0,y_0,z_0)$ is a point on the surface. We want to write an equation of the plane that is tangent to the surface at point *P*.

Recall: a plane passing through the point $P(x_0,y_0,z_0)$ can be represented by the equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, where $\langle a,b,c\rangle$ is a vector normal to the plane.

Tangent Plane and Normal Line to a Surface:

Suppose a surface *S* is given by $F(x,y,z)$. If *F* is differentiable at the point $P(x_0,y_0,z_0)$ on the surface, and if $\nabla F(x_0,y_0,z_0)\neq \mathbf{0}$, then

1. The plane through *P* that is normal to $\nabla F(x_0,y_0,z_0)$ is called the *tangent plane* to *S* at *P*.

2. The line through *P* having the direction of $\nabla F(x_0,y_0,z_0)$ is called the *normal line* to *S* at *P*.

3. The equation of the tangent plane to *S* at *P* is

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.$$

Example 1: Find equations for the tangent plane and the normal line to the surface $x^2 + y^2 + z^2 = 9$ at the point $(1,2,2)$.

Example 2: Find equations for the tangent plane and the normal line to the surface $z = x^2 - 2xy + y^2$ at the point $(1,2,1)$.

Example 3: Find an equation for the tangent plane to the surface given by the function $f(x,y) = \ln(x^2 + y^2)$, at the point $(1,2)$.

Angle of inclination:

Definition: The angle of inclination of a plane S is the angle θ, with $0 \le \theta \le \frac{\pi}{2}$, between the plane S and the xy-plane.

If a plane has normal vector $\mathbf{n}$, the plane's angle of inclination satisfies the equation

$$\cos\theta = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|\|\mathbf{k}\|} = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|}.$$

Example 4: Find the angle of inclination of the plane that is tangent to the surface given by $2xy - z^3 = 0$ at the point $(2,2,2)$.

Example 5: Find the point(s) on the surface $z = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$ at which the tangent plane is horizontal.