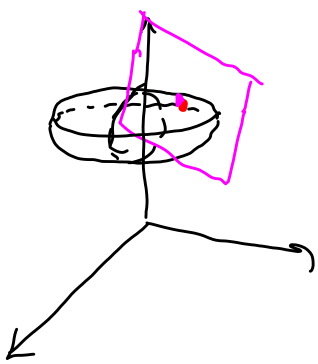
13.7: Tangent Planes and Normal Lines

Suppose a surface in $\mathbb{R}^3$ is represented by the equation $F(x, y, z) = 0$, and that $P(x_0, y_0, z_0)$ is a point on the surface. We want to write an equation of the plane that is tangent to the surface at point P.



Let P be a point on surface described by $F(x,y,z)=0$

Suppose $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ describes a curve C that is completely contained in the surface, and so that C goes thru P.

Then $F(x(t), y(t), z(t)) = 0$ for every t.
(Because $F(x,y,z)=0$ on the surface)

$\frac{d}{dt}\left(F(x(t), y(t), z(t))\right) = \frac{d}{dt}(0)$

$F'(t) = 0$

$F'(t) = F_x(x,y,z)\,x'(t) + F_y(x,y,z)\,y'(t) + F_z(x,y,z)\,z'(t) = 0$ (chain rule)

$= \nabla F(x,y,z) \cdot \vec{r}\,'(t) = 0$

Recall: a plane passing through the point $P(x_0, y_0, z_0)$ can be represented by the equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, where $\langle a, b, c\rangle$ is a vector normal to the plane.

$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle$

so gradient is orthogonal to every curve $\vec{r}(t)$.
so ∇F is the normal vector to the tangent plane

Tangent Plane and Normal Line to a Surface:

Suppose a surface S is given by $F(x, y, z)$. If F is differentiable at the point $P(x_0, y_0, z_0)$ on the surface, and if $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the *tangent plane* to S at P.

2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the *normal line* to S at P.

3. The equation of the tangent plane to S at P is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Note: Normal line:

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} \text{parametric equations}$$

solve for t to get the symmetric equations: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

Example 1: Find equations for the tangent plane and the normal line to the surface $x^2+y^2+z^2=9$ at the point $(1,2,2)$.

$x^2+y^2+z^2-9=0$

$F(x,y,z)=x^2+y^2+z^2-9$

$\nabla F(x,y,z) = \langle F_x, F_y, F_z\rangle = \langle 2x, 2y, 2z\rangle$

$\nabla F(1,2,2) = \langle 2,4,4\rangle$ (Normal vector for plane)

Plane: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ for normal vector $\langle a,b,c\rangle$

$\boxed{2(x-1)+4(y-2)+4(z-2)=0}$ ← eqn of plane.

$2x-2+4y-8+4z-8=0$

$2x+4y+4z-18=0$

Divide by 2: $\boxed{x+2y+2z-9=0}$

Parametric eqns for normal line:

$$\boxed{\begin{aligned} x &= 1t+1 \\ y &= 2t+2 \\ z &= 2t+2 \end{aligned}}$$

or, using $\vec{N} = \langle 2,4,4\rangle$

$$\boxed{\begin{aligned} x &= 2t+1 \\ y &= 4t+2 \\ z &= 4t+2 \end{aligned}}$$

Example 2: Find equations for the tangent plane and the normal line to the surface $z=x^2-2xy+y^2$ at the point $(1,2,1)$.

$F(x,y,z)=x^2-2xy+y^2-z=0$

$\nabla F(x,y,z) = \langle 2x-2y, -2x+2y, -1\rangle$

$\nabla F(1,2,1) = \langle 2(1)-2(2), -2(1)+2(2), -1\rangle = \langle -2, 2, -1\rangle$ (normal vector for tangent plane)

Tangent plane: $-2(x-1)+2(y-2)-1(z-1)=0$

$-2x+2+2y-4-z+1=0$

$\boxed{-2x+2y-z-1=0}$ tangent plane

Angle of Inclination

$\cos\theta = \dfrac{|\vec{n}\cdot\vec{k}|}{\|\vec{n}\|} = \dfrac{\langle -2,2,-1\rangle\cdot\langle 0,0,1\rangle}{\sqrt{9}} = \dfrac{1}{3}$

$\theta \approx \cos^{-1}\left(\frac{1}{3}\right) \approx 70.5^\circ$

Normal line: Parametric eqns:

$$\boxed{\begin{aligned} x &= -2t+1 \\ y &= 2t+2 \\ z &= -t+1 \end{aligned}}$$

Symmetric eqns:

$$\boxed{\frac{x-1}{-2} = \frac{y-2}{2} = \frac{z-1}{-1}}$$

Example 3: Find an equation for the tangent plane to the surface given by the function $f(x,y)=\ln(x^2+y^2)$, at the point $(1,2)$.

At $(1,2)$, $z=f(1,2)=\ln(1^2+2^2)=\ln 5$

$z=\ln(x^2+y^2)$

$F(x,y,z)=\ln(x^2+y^2)-z=0$. Point on surface is $(x_0,y_0,z_0)=(1,2,\ln 5)$

$\nabla F(x,y,z) = \langle F_x, F_y, F_z\rangle = \left\langle \frac{1}{x^2+y^2}(2x), \frac{1}{x^2+y^2}(2y), -1\right\rangle$

$= \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}, -1\right\rangle$

$\nabla f(1,2,\ln 5) = \left\langle \frac{2(1)}{1^2+2^2}, \frac{2(2)}{1^2+2^2}, -1\right\rangle = \left\langle \frac{2}{5}, \frac{4}{5}, -1\right\rangle$

Tangent Plane: $\frac{2}{5}(x-1)+\frac{4}{5}(y-2)-1(z-\ln 5)=0$

$\frac{2}{5}x-\frac{2}{5}+\frac{4}{5}y-\frac{8}{5}-z+\ln 5=0$

Tangent Plane: $\boxed{\frac{2}{5}x+\frac{4}{5}y-z-2+\ln 5=0}$

Parametric eqns of normal line

$$\boxed{\begin{aligned} x &= \tfrac{2}{5}t+1 \\ y &= \tfrac{4}{5}t+2 \\ z &= -t+\ln 5 \end{aligned}}$$

Angle of inclination:

Definition: The angle of inclination of a plane S is the angle θ, with $0 \le \theta \le \frac{\pi}{2}$, between the plane S and the xy-plane.

If a plane has normal vector $\mathbf{n}$, the plane's angle of inclination satisfies the equation

$$\cos\theta = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|\|\mathbf{k}\|} = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|}.$$

Example 4: Find the angle of inclination of the plane that is tangent to the surface given by $2xy - z^3 = 0$ at the point $(2,2,2)$.

Find a normal vector for the tangent plane:

$F(x,y,z) = 2xy - z^3$

$\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle = \langle 2y, 2x, -3z^2 \rangle$

Normal vector is $\nabla F(2,2,2) = \langle 2(2), 2(2), -3(2)^2 \rangle = \langle 4, 4, -12 \rangle$

You could use any scalar multiple of this vector. [could use $\langle 1,1,-3\rangle$]

For $\vec{n} = \langle 1,1,-3 \rangle$, $\|\vec{n}\| = \sqrt{1^2+1^2+3^2} = \sqrt{11}$

$\cos\theta = \frac{|\vec{n}\cdot\vec{k}|}{\|\vec{n}\|} = \frac{|\langle 1,1,-3\rangle \cdot \langle 0,0,1\rangle|}{\sqrt{11}}$

$= |0+0+-3| \frac{1}{\sqrt{11}} = \frac{3}{\sqrt{11}}$

$\theta = \cos^{-1}\left(\frac{3}{\sqrt{11}}\right)$

$\approx \boxed{25.24^\circ}$

Example 5: Find the point(s) on the surface $z = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$ at which the tangent plane is horizontal.

$F(x,y,z) = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4 - z = 0$

For the tangent plane to be horizontal, we need $\theta = 0$. So $\cos\theta = 1$

(Really, just think of the normal vectors as being parallel to each other. So we want $\vec{n} \parallel \vec{k}$

$\nabla F(x,y,z) = \langle 8x + 4y + 8, 4x - 4y - 5, -1 \rangle$

For $\vec{n} \parallel \vec{k}$ ($\vec{n}$ parallel to $\vec{k}$), we need $F_x = 0$ and $F_y = 0$.

$8x + 4y + 8 = 0$

$4x - 4y - 5 = 0$

Add eqns: $12x + 3 = 0$

$12x = -3$

$x = -\frac{3}{12} = -\frac{1}{4}$

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Ex 5 cont'd:

Either put $x = -\frac{1}{4}$ into one of the original eqns, or do elimination again:

$$\begin{array}{lcl} 8x + 4y + 8 = 0 & \longrightarrow & 8x + 4y + 8 = 0 \\ 4x - 4y - 5 = 0 & \xrightarrow{(-2)} & -8x + 8y + 10 = 0 \end{array}$$

Add eqns: $12y + 18 = 0$

$12y = -18$

$y = \frac{-18}{12} = \frac{-3}{2}$

Tangent plane is horizontal at the point $\left(-\frac{1}{4}, -\frac{3}{2}\right)$ in $\mathbb{R}^2$.

To get the point on the surface, we need to find z:

$$z = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$$

$$z = 4\left(-\frac{1}{4}\right)^2 + 4\left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right) - 2\left(-\frac{3}{2}\right)^2 + 8\left(-\frac{1}{4}\right) - 5\left(-\frac{3}{2}\right) - 4$$

$$= \frac{4}{16} + \frac{12}{8} - 2\left(+\frac{9}{4}\right) - \frac{8}{4} + \frac{15}{2} - 4$$

$$= \frac{1}{4} + \frac{6}{4} - \frac{18}{4} - \frac{8}{4} + \frac{30}{4} - \frac{16}{4}$$

$$= -\frac{5}{4}$$

So the tangent plane is horizontal at the point $\left(-\frac{1}{4}, -\frac{3}{2}, -\frac{5}{4}\right)$.