13.8: Extrema of Functions of Two Variables

Definition:

Extreme Values:

The values $f(a,b)$ and $f(c,d)$ are called the *minimum and maximum values*, respectively, of f in a region *R* if $f(a,b) \le f(x,y) \le f(c,d)$ for every (x,y) in *R*.

(For clarity, sometimes these are called *absolute* or *global* extreme values, to distinguish them from relative (local) extreme values.)

Relative Extreme Values:

A function $f(x,y)$ has a *relative (local) maximum* at (x_0,y_0) if $f(x,y) \le f(x_0,y_0)$ for all points (x,y) in an open disk containing (x_0,y_0). The value $f(x_0,y_0)$ is called a *relative maximum* (or *local maximum*) of *f*.

A function $f(x,y)$ has a *relative (local) minimum* at (x_0,y_0) if $f(x,y) \ge f(x_0,y_0)$ for all points (x,y) in an open disk containing (x_0,y_0). The value $f(x_0,y_0)$ is called a *relative minimum* (or *local minimum*) of *f*.

Extreme Value Theorem:

Suppose $f(x,y)$ is a continuous function defined on a closed and bounded region *R* in the *xy*-plane. Then,

1. There is at least one point in *R* at which f takes on a minimum value.
2. There is at least one point in *R* at which f takes on a maximum value.

Definition: Critical Point

Let *f* be defined on an open region (x_0,y_0). The point (x_0,y_0) is a *critical point* if one of the following statements is true.

1. $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$

2. $f_x(x_0,y_0)$ does not exist, or $f_y(x_0,y_0)$ does not exist.

Example 1: Find the critical points for the function $f(x, y) = (x^2 + y^2)^{2/3}$.

Theorem:

If f has a relative minimum or relative maximum at the point (x_0, y_0), then (x_0, y_0) must be a critical point of f.

This means that if both of the first-order partial derivatives exist at a relative extremum, then the tangent plane must be horizontal.

Note: As with functions of one variable, not every critical point yields a relative extremum.

Example 2: Find the critical points and relative extrema for the function.

$$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20$$

Example 3: Find the critical points and relative extrema for the function.

$$g(x,y) = 1 - \sqrt[3]{x^2 + y^2}$$

Example 4: Find the critical points and relative extrema for the function.

$$h(x,y) = 2x^2 - 3y^2$$

Example 5: Find the critical points and relative extrema for the function.

$$h(x,y)=2x^2-3y^2$$

Theorem: Second Partials Test

Suppose the second partial derivatives of $f(x,y)$ are continuous on an open region containing the point (a,b).

Suppose also that $f_x(a,b)=0$ and $f_y(a,b)=0$.

Define $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$. Then,

1. If $D(a,b)>0$ and $f_{xx}(a,b)>0$, then f has a relative minimum at (a,b).

2. If $D(a,b)>0$ and $f_{xx}(a,b)<0$, then f has a relative maximum at (a,b).

3. If $D(a,b)<0$, then there is a saddle point (and thus, no relative extremum) at (a,b).

4. If $D(a,b)=0$, then the second partials test is inconclusive.

Note: D can be written as a determinant:

$$D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=f_{xx}f_{yy}-f_{xy}f_{yx}=f_{xx}f_{yy}-\left(f_{xy}\right)^2$$

Example 6: Find and classify the critical points of $f(x,y) = -x^3 + 4xy - 2y^2 + 1$.

Example 7: Find and classify the critical points of $f(x,y) = 3x^2 + 12xy + 2y^2 + 6y + 5$.

Example 8: Find and classify the critical points of $f(x,y) = x^3 + 2xy^2 + 5x^2 + 4y^2 + 3$.

Example 9: Find and classify the critical points of $f(x,y) = 2xy - \frac{1}{2}x^4 - -\frac{1}{2}y^4 + 1$.

Example 10: Find the absolute extrema of $f(x, y) = 3x^2 + 2y^2 - 4y$ over the region bounded by the graphs of $y = x^2$ and $y = 4$.