13.8: Extrema of Functions of Two Variables

Definition:

Extreme Values:

The values $f(a,b)$ and $f(c,d)$ are called the *minimum and maximum values*, respectively, of f in a region *R* if $f(a,b) \le f(x,y) \le f(c,d)$ for every (x,y) in *R*.

(For clarity, sometimes these are called *absolute* or *global* extreme values, to distinguish them from relative (local) extreme values.)

Relative Extreme Values:

A function $f(x,y)$ has a *relative (local) maximum* at (x_0, y_0) if $f(x,y) \le f(x_0, y_0)$ for all points (x,y) in an open disk containing (x_0, y_0). The value $f(x_0, y_0)$ is called a *relative maximum* (or *local maximum*) of *f*.

A function $f(x,y)$ has a *relative (local) minimum* at (x_0, y_0) if $f(x,y) \ge f(x_0, y_0)$ for all points (x,y) in an open disk containing (x_0, y_0). The value $f(x_0, y_0)$ is called a *relative minimum* (or *local minimum*) of *f*.

Extreme Value Theorem:

Suppose $f(x,y)$ is a continuous function defined on a closed and bounded region *R* in the *xy*-plane. Then,

1. There is at least one point in *R* at which *f* takes on a minimum value. AND
2. There is at least one point in *R* at which *f* takes on a maximum value.

Definition: Critical Point

Let *f* be defined on an open region (x_0, y_0). The point (x_0, y_0) is a *critical point* if one of the following statements is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

OR

2. $f_x(x_0, y_0)$ does not exist, or $f_y(x_0, y_0)$ does not exist.

Example 1: Find the critical points for the function $f(x, y) = (x^2 + y^2)^{2/3}$.

$f(x,y) = (x^2+y^2)^{2/3}$

$f_x(x,y) = \frac{2}{3}(x^2+y^2)^{-1/3}(2x) = \frac{4x}{3\sqrt[3]{x^2+y^2}}$

$f_y(x,y) = \frac{2}{3}(x^2+y^2)^{-1/3}(2y) = \frac{4y}{3\sqrt[3]{x^2+y^2}}$

f_x and f_y do not exist at $(0,0)$. Thus $(0,0)$ is a critical point.

> Theorem:
>
> If *f* has a relative minimum or relative maximum at the point (x_0, y_0), then (x_0, y_0) must be a critical point of *f*.

This means that if both of the first-order partial derivatives exist at a relative extremum, then the tangent plane must be horizontal.

Note: As with functions of one variable, not every critical point yields a relative extremum.

Example 2: Find the critical points and relative extrema for the function.

$$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20$$

$\nabla f(x,y) = \langle 4x+8, 2y-6 \rangle$

For critical points, set $f_x = 0$ and $f_y = 0$:

2 eqns, 2 variables $\begin{cases} 4x+8=0 \\ 2y-6=0 \end{cases}$

$4x+8=0$
$4x=-8$
$x=-2$

$2y-6=0$
$2y=6$
$y=3$

$\boxed{\text{So the point } (-2,3) \text{ is the only critical point.}}$

$f(x,y) = 2x^2+y^2+8x-6y+20$

Complete the square: $2x^2+8x+y^2-6y+20 = f(x,y)$

$f(x,y) = 2(x^2+4x+4) + (y^2-6y+9) + 20 - 8 - 9$

$f(x,y) = 2(x+2)^2 + (y-3)^2 + 3$

At $x=-2, y=3$,
$f(-2,3) = 3$

For all other (x,y),
$f(x,y) > 3$.

So, f has a relative minimum at $(-2,3)$.

$\boxed{\text{Relative minimum is } f(-2,3)=3}$

Example 3: Find the critical points and relative extrema for the function.

$$g(x,y) = 1 - \sqrt[3]{x^2+y^2} = 1 - (x^2+y^2)^{1/3}$$

$$g_x(x,y) = -\frac{1}{3}(x^2+y^2)^{-2/3}(2x) = -\frac{2x}{3(x^2+y^2)^{2/3}} = -\frac{2x}{3\sqrt[3]{(x^2+y^2)^2}}$$

$$g_y(x,y) = -\frac{1}{3}(x^2+y^2)^{-2/3}(2y) = -\frac{2y}{3(x^2+y^2)^{2/3}} = -\frac{2y}{3\sqrt[3]{(x^2+y^2)^2}}$$

$\boxed{\text{Critical point(s): } (0,0)}$ because the partial derivative aren't defined at $(0,0)$.

$$g(0,0) = 1 - \sqrt[3]{0^2+0^2}$$
$$= 1 - 0 = 1$$

For other (x,y) what happens to the value of g?

For all other (x,y), $g(x,y) < 1$.

$\boxed{\text{So, } g \text{ has a relative maximum } g(0,0) = 1.}$

(also the absolute max)

Example 4: Find the critical points and relative extrema for the function.

$$h(x,y) = 2x^2 - 3y^2$$

$h_x(x,y) = 4x$

$h_y(x,y) = -6y$

Setting $h_x = 0$ and $h_y = 0$:

$4x = 0 \qquad -6y = 0$

$x = 0 \qquad y = 0$

$\boxed{\text{Critical Point: } (0,0)}$

We know $h(0,0) = 0$.

Along the x-axis $(y=0)$, we have $h(x,0) = 2x^2 - 3(0)^2 = 2x^2 \geq 0$ for all x.

$2x^2 > 0$ for all $x \neq 0$

Along the y-axis $(x=0)$, we have $h(0,y) = 2(0)^2 - 3y^2 = -3y^2 \leq 0$ for all y

$-3y^2 < 0$ for all $y \neq 0$

So, h does not have a relative extreme at $(0,0)$

$\boxed{\text{No relative extrema}}$

Example 5: Find the critical points and relative extrema for the function.

$h(x, y) = 2x^2 - 3y^2$

Theorem: Second Partials Test

Suppose the second partial derivatives of $f(x, y)$ are continuous on an open region containing the point (a,b).

Suppose also that $f_x(a,b) = 0$ and $f_y(a,b) = 0$. (so (a,b) is a critical point)

Define $D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b) \right]^2$. Then,

1. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a relative minimum at (a,b).
2. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a relative maximum at (a,b).
3. If $D(a,b) < 0$, then there is a saddle point (and thus, no relative extremum) at (a,b).
4. If $D(a,b) = 0$, then the second partials test is inconclusive.

Note: D can be written as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy} f_{yx} = f_{xx} f_{yy} - \left(f_{xy} \right)^2$$

$f_{xy} = f_{yx}$ for functions with continuous partial derivatives

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yx} \end{vmatrix} \begin{array}{l} \leftarrow x \text{ Row 1} \\ \leftarrow y \text{ row 2} \end{array}$$

Example 6: Find and classify the critical points of $f(x, y) = -x^3 + 4xy - 2y^2 + 1$.

$f_x(x,y) = -3x^2 + 4y$

$f_y(x,y) = 4x - 4y$

$-3x^2 + 4y = 0$

$4x - 4y = 0 \Rightarrow 4x = 4y \Rightarrow y = x$

Substitute $y = x$ into $-3x^2 + 4y = 0$

$-3x^2 + 4x = 0$

$x(-3x+4) = 0$

$x = 0$ | $-3x+4=0$, $-3x = -4$, $x = \frac{4}{3}$

$x = 0$ | $4x - 4y = 0$

$4(0) - 4y = 0$

$-4y = 0$

$y = 0$ Critical Pt $(0,0)$

$x = \frac{4}{3}$ | $4x - 4y = 0$

$4\left(\frac{4}{3}\right) - 4y = 0$

$\frac{16}{3} = 4y$

$16 = 12y$

$\frac{16}{12} = y$

$y = \frac{4}{3}$ Critical Pt $\left(\frac{4}{3}, \frac{4}{3}\right)$

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Example 7: Find and classify the critical points of $f(x, y) = 3x^2 + 12xy + 2y^2 + 6y + 5$.

$f_x(x,y) = 6x + 12y$

$f_y(x,y) = 12x + 4y + 6$

$f_{xx}(x,y) = 6$

$f_{xy}(x,y) = 12$

$f_{yy}(x,y) = 4$

$f_{yx}(x,y) = 12$

Set $f_x = f_y = 0$:

$6x + 12y = 0 \xrightarrow{(-2)} -12x - 24y = 0$

$12x + 4y + 6 = 0 \longrightarrow 12x + 4y + 6 = 0$

$-20y + 6 = 0$

$-20y = -6$

$y = -\frac{6}{-20} = \frac{3}{10}$

$6x + 12y = 0 \Rightarrow 6x + 12\left(\frac{3}{10}\right) = 0$

$6x = -\frac{36}{10}$

$x = -\frac{\cancel{36}^{6}}{10}\left(\frac{1}{\cancel{6}_{1}}\right) = -\frac{6}{10} = -\frac{3}{5}$

Critical Pt: $\left(-\frac{3}{5}, \frac{3}{10}\right)$

$D = f_{xx} f_{yy} - (f_{yx})^2$

$= 6(4) - 12^2 = 24 - 144 = -120 < 0$

Saddle Point (no relative extreme): $\left(-\frac{3}{5}, \frac{3}{10}\right)$

Ex 6 cont'd: $\boxed{\text{Critical Pts } (0,0) \text{ and } (\tfrac{4}{3},\tfrac{4}{3})}$

$f_x(x,y) = -3x^2 + 4y$ $\qquad$ $f_y(x,y) = 4x - 4y$

$f_{xx}(x,y) = -6x$ $\qquad$ $f_{yx}(x,y) = 4$

$f_{xy}(x,y) = 4$ $\qquad \overset{=}{\checkmark}$ $\qquad$ $f_{yy} = -4$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -6x & 4 \\ 4 & -4 \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$
$$= -6x(-4) - 4^2$$
$$= 24x - 16$$

$D(0,0) = 24(0) - 16 = -16 < 0$

So there is a $\boxed{\text{saddle point at } (0,0).}$

$D(\tfrac{4}{3},\tfrac{4}{3}) = 24(\tfrac{4}{3}) - 16 = 32 - 16 = 16 > 0$

$f_{xx}(x,y) = -6x$

$f_{xx}(\tfrac{4}{3},\tfrac{4}{3}) = -6(\tfrac{4}{3}) = -8 < 0$.

$\boxed{\text{So } f \text{ has a relative max at } (\tfrac{4}{3},\tfrac{4}{3}).}$

Ex 8 cont'd:

Critical Pts: $(0,0), (-\tfrac{10}{3},0), (-2,2), (-2,-2)$

$f_{xx}(x,y) = 6x + 10$

$f_{yy}(x,y) = 4x + 8$

$f_{xy}(x,y) = 4y$ $\quad = \checkmark$

$f_{yx}(x,y) = 4y$

$D(-2,-2) = 24(-2)^2 + 88(-2) + 80 - 16(-2)^2$
$= -64 < 0$

$\boxed{\text{Saddle pt at } (-2,-2)}$

$D(-2,2) = 24(-2)^2 + 88(-2) + 80 - 16(2)^2 = -64$

$\boxed{\text{Saddle Pt at } (-2,2)}$

$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$
$= (6x+10)(4x+8) - (4y)^2$
$= 24x^2 + 88x + 80 - 16y^2$

$D(0,0) = 80 > 0$

$f_{xx}(0,0) = 6(0) + 10 = 10 > 0$

$\boxed{\text{Relative min at } (0,0)}$

$D(-\tfrac{10}{3},0) = 24(-\tfrac{10}{3})^2 + 88(-\tfrac{10}{3}) + 80 - 16(0)^2$
$= \frac{2400}{9} - \frac{880}{3} + 80$
$= 53.3 > 0$

$f_{xx}(-\tfrac{10}{3},0) = 6(-\tfrac{10}{3}) + 10 = -20 + 10 = -10 < 0$

$\boxed{\text{Relative Max at } (-\tfrac{10}{3},0)}$

Example 8: Find and classify the critical points of $f(x, y) = x^3 + 2xy^2 + 5x^2 + 4y^2 + 3$.

$f_x(x,y) = 3x^2 + 2y^2 + 10x$

$f_y(x,y) = 4xy + 8y$

$f_{xx}(x,y) = 6x + 10$

$f_{yy}(x,y) = 4x + 8$

$f_{xy}(x,y) = 4y$ $= \checkmark$

$f_{yx}(x,y) = 4y$

Set $f_x = f_y = 0$:

$3x^2 + 2y^2 + 10x = 0$

$4xy + 8y = 0 \Rightarrow 4y(x+2=0) \Rightarrow y=0, x=-2$

$\boxed{y=0}$ $3x^2+2y^2+10x=0$

$3x^2+2(0)^2+10x=0$

$3x^2+10x=0$

$x(3x+10)=0$

$x=0, x=-\frac{10}{3}$

Critical Pts: $(0,0), (-\frac{10}{3}, 0)$

$\boxed{x=-2}$ $3x^2+2y^2+10x=0$

$3(-2)^2+2y^2+10(-2)=0$

$12+2y^2-20=0$

$2y^2-8=0$

$2y^2=8$

$y^2=4$

$y=\pm 2$

Critical Pts: $(-2,2), (-2,-2)$

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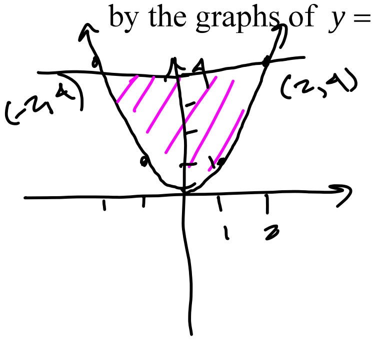
Example 9: Find and classify the critical points of $f(x, y) = 2xy - \frac{1}{2}x^4 - -\frac{1}{2}y^4 + 1$.

Example 10: Find the absolute extrema of $f(x,y) = 3x^2 + 2y^2 - 4y$ over the region bounded by the graphs of $y = x^2$ and $y = 4$.



Find intersection pts: set y's equal:

$x^2 = 4 \Rightarrow x = \pm 2$

$f_x(x,y) = 6x$ $\qquad f_{xx}(x,y) = 6$

$f_y(x,y) = 4y - 4$ $\qquad f_{yy}(x,y) = 4$

$f_{xy}(x,y) = 0$

$f_{yx}(x,y) = 0$ $\quad = \checkmark$

Find critical pts: Set $f_x = f_y = 0$

$6x = 0 \Rightarrow x = 0$

$4y - 4 = 0 \Rightarrow y = 1$

Critical Point: $(0,1)$. Is it in region? Yes.

At $(0,1)$: $\left.\begin{array}{l} f_{xx}(0,1) = 6 \\ f_{yy}(0,1) = 4 \\ f_{xy}(0,1) = 0 \end{array}\right\}$ $\quad D(0,1) = 6(4) - 0^2 = 24 > 0$

$f_{xx}(0,1) = 6 > 0$

Relative minimum at $(0,1)$

$z = f(0,1) = 3(0)^2 + 2(1)^2 - 4(1)$

$= 0 + 2 - 4 = -2$

<u>Analyze Boundaries:</u> $\quad f(x,y) = 3x^2 + 2y^2 - 4y$

<u>Along the line $y=4$:</u> $f(x,4) = 3x^2 + 2(4)^2 - 4(4)$

$z = 3x^2 + 16$

what are min/max of z along this boundary?

min along $y=4$: 16 (when $x=0$)

max along $y=4$: occurs when $x = \pm 2$

$x = \pm 2 \Rightarrow \quad z = 3x^2 + 16 = 3(\pm 2)^2 + 16$

$= 28$

<u>Along the parabola $y = x^2$:</u> $f(x,y) = 3x^2 + 2y^2 - 4y$

$f(x, x^2) = 3x^2 + 2(x^2)^2 - 4(x^2)$

$= 2x^4 - x^2$

We need to find the max/min of $z_2(x) = 2x^4 - x^2$

$z_2'(x) = 8x^3 - 2x$

$= 2x(4x^2 - 1)$

Setting $z_2'(x) = 0$: $x = 0$, $4x^2 = 1$

$x^2 = \frac{1}{4}$

$x = \pm \frac{1}{2}$

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$z_2(x) = 2x^4 - x^2$ along parabolic boundary

$z_2(0) = 2(0)^4 - 0^2 = 0$

$z_2(\frac{1}{2}) = 2(\frac{1}{2})^4 - (\frac{1}{2})^2 = 2(\frac{1}{16}) - \frac{1}{4} = \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$

$z_2(-\frac{1}{2}) = 2(-\frac{1}{2})^4 - (-\frac{1}{2})^2$ (same)

$= -\frac{1}{8}$

$z_2(\pm 2) = 2(\pm 2)^4 - (\pm 2)^2 = 2(16) - 4 = 32 - 4 = 28$

Along parabolic boundary,

the min is $-\frac{1}{8}$

the max is 28

Along top boundary, min was 16

max was 28

In interior: Relative min was $f(0,1) = -2$

$$\boxed{\begin{array}{l}\text{Absolute Minimum is } f(0,1) = -2 \\ \text{Absolute maximum is } f(-2,4) = f(2,4) = 28\end{array}}$$