13.9: Applications of Extrema

Example 1: Find three positive numbers x, y, and z such that the sum is 32 and $P = xy^2z$ is at a maximum.

$P(x,y,z) = xy^2z$

Sum $= 32$: $x+y+z=32$

Write P in terms of x and y only.

$x+y+z=32 \Rightarrow z = 32-x-y$ Substitute into P:

$P = xy^2z \Rightarrow P = xy^2(32-x-y)$

$\hat{P}(x,y) = 32xy^2 - x^2y^2 - xy^3$

$\hat{P}_x(x,y) = 32y^2 - 2xy^2 - y^3$

$\hat{P}_y(x,y) = 64xy - 2x^2y - 3xy^2$

$\hat{P}_{xx}(x,y) = -2y^2$

$\hat{P}_{yy}(x,y) = 64x - 2x^2 - 6xy$

$\hat{P}_{xy}(x,y) = 64y - 4xy - 3y^2$ ✓ $=$

$\hat{P}_{yx}(x,y) = 64y - 4xy - 3y^2$

Set $\hat{P}_x = \hat{P}_y = 0$:

$32y^2 - 2xy^2 - y^3 = 0$

$64xy - 2x^2y - 3xy^2 = 0$

$32y^2 - 2xy^2 - y^3 = 0$

$y^2(32-2x-y) = 0$

$y^2 = 0$, $32-2x-y=0$

Throw out $y=0$ (problem specified positive #s)

$64xy - 2x^2y - 3xy^2 = 0$

$xy(64-2x-3y) = 0$

$x=0$ $y=0$ (Throw out)

$\Leftarrow$

$32-2x-y=0$

$64-2x-3y=0$

Subtract eqns: $-32+2y=0$

$2y = 32$

$y = 16$

Put $y=16$ into $32-2x-y=0$:

$32-2x-16=0$

$16-2x=0$

$16=2x$

$x=8$

Only critical point: $(8,16)$

$\hat{P}_{xx}(x,y) = -2y^2 \Rightarrow \hat{P}_{xx}(8,16) = -2(16)^2 = -512$

$\hat{P}_{yy}(x,y) = 64x - 2x^2y - 6xy$

$P_{yy}(8,16) = 64(8) - 2(8)^2(16) - 6(8)(16) = -384$

$P_{xy}(x,y) = 64y - 4xy - 3y^2$

$P_{xy}(8,16) = 64(16) - 4(8)(16) - 3(16)^2$

$= -256$

$D = (-512)(-384) - (-256)^2$

$131072 > 0$

$\hat{P}_{xx}(8,16) = -512 < 0 \Rightarrow$ rel max at $(8,16)$

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$x=8, y=16 \Rightarrow z = 32-x-y = 32-8-16 = 32-24 = 8$

$P(8,16,8) = 8(16)^2(8) = 64(256) = 16384$.

$\boxed{\text{The numbers are } x=8, y=16, z=8}$ (write answer in a complete sentence)

Example 2: Melika Candle Company manufactures candles at two locations. At Location A, the cost of producing x units is $C_A = 0.02x^2 + 4x + 500$. At Location B, the cost of producing x units is $C_B = 0.05x^2 + 4x + 275$. The candles sell for \$15 per unit. Find the quantity that should be produced at each location to maximize profit.

Maximize: P = Profit.

need to find a function for this

Profit = Revenue − Production Cost

x = number of candles made at Location A

y = number of candles made at Location B

Revenue Location A: $15x$

Cost for Location A: $0.02x^2 + 4x + 500$

Revenue Location B: $15y$

Cost for Location B: $0.05y^2 + 4y + 275$

$$P(x,y) = 15x + 15y - (0.02x^2 + 4x + 500) - (0.05y^2 + 4y + 275)$$

$$= 11x - 0.02x^2 + 11y - 0.05y^2 - 775$$

$P_x(x,y) = 11 - 0.04x$
$P_y(x,y) = 11 - 0.10y$ } set these $=0$:

$11 - 0.04x = 0$

$11 = 0.04x$

$275 = x$

$11 - 0.10y = 0$

$11 = 0.10y$

$110 = y$

$P_{xx}(x,y) = -0.04$

$P_{yy}(x,y) = -0.10$

$P_{xy}(x,y) = P_{yx}(x,y) = 0$

Critical Pt: $(x,y) = (275, 110)$

$D(275,110) = -0.04(-0.10) - 0^2 = 0.004 > 0$

$P_{xx}(275,110) = -0.04 < 0$ We have a relative max

$\boxed{\text{The company should make 275 candles at Location A, and 110 candles at Location B.}}$

Example 3: Find the distance from the point $(1,0,-2)$ to the plane $x+2y+z=4$.

Minimize: Distance from $(1,0,-2)$ to a point on the plane.

Distance between $(1,0,-2)$ and $P(x,y,z)$ on plane $x+2y+z=4$ is:

$$d=\sqrt{(x-1)^2+(y-0)^2+(z-(-2))^2}$$

want to write it as a function of 2 variables.

$$d=\sqrt{(x-1)^2+y^2+(z+2)^2}$$

Solve plane eqn for z:

substitute $z=4-x-2y$

$$d=\sqrt{(x-1)^2+y^2+(4-x-2y+2)^2}$$

Minimizing d is equivalent to minimizing $M=d^2=(x-1)^2+y^2+(6-x-2y)^2$

$$M_x(x,y)=2(x-1)(1)+2(6-x-2y)(-1)$$
$$=2x-2-12+2x+4y = 4x+4y-14$$

$$M_y(x,y)=2y+2(6-x-2y)(-2)$$
$$=2y-24+4x+8y$$
$$=4x+10y-24$$

Set $M_x=M_y=0$: $4x+4y-14=0$
$4x+10y-24=0$

Subtract eqns: $-6y+10=0$
$-6y=-10$
$y=\frac{-10}{-6}=\frac{5}{3}$

$M_{xx}(x,y)=4$
$M_{yy}(x,y)=10$
$M_{xy}(x,y)=4$
$M_{yx}(x,y)=4$ ✓

$4x+4y-14=0$
$y=\frac{5}{3} \Rightarrow 4x+4\left(\frac{5}{3}\right)-14=0$
$4x+\frac{20}{3}-\frac{42}{3}=0$
$4x-\frac{22}{3}=0$
$4x=\frac{22}{3}$
$x=\frac{22}{3}\left(\frac{1}{4}\right)=\frac{22}{12}=\frac{11}{6}$

Critical pt: $\left(\frac{11}{6},\frac{5}{3}\right)$

$D(x,y)=4(10)-4^2$
$D\left(\frac{11}{6},\frac{5}{3}\right)=40-16=24>0$

Check sign of M_{xx}: $M_{xx}=4>0$

relative minimum (absolute)

Find the z: $z=4-x-2y$

$x=\frac{11}{6}, y=\frac{5}{3} \Rightarrow z=4-\frac{11}{6}-2\left(\frac{5}{3}\right)$
$=\frac{24}{6}-\frac{11}{6}-\frac{20}{6}=-\frac{7}{6}$

$$\text{distance}=\sqrt{\left(1-\tfrac{11}{6}\right)^2+\left(0-\tfrac{5}{3}\right)^2+\left(-2-\left(-\tfrac{7}{6}\right)\right)^2}$$
$$=\sqrt{\left(-\tfrac{5}{6}\right)^2+\tfrac{25}{9}+\left(-\tfrac{12}{6}+\tfrac{7}{6}\right)^2}$$

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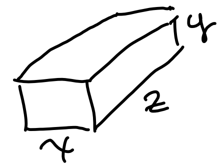
$$d = \sqrt{\frac{25}{36} + \frac{25}{9} + \left(\frac{-5}{6}\right)^2} = \sqrt{\frac{25}{36} + \frac{25}{9} + \frac{25}{36}} = \sqrt{\frac{25}{6}}$$

$$= \frac{5}{\sqrt{6}} = \boxed{\frac{5\sqrt{6}}{6}} \text{ min. distance}$$

Example 4: A rectangular box without a lid is to be made from 12 square meters of cardboard. Find the maximum volume of such a box.



Maximize: Volume $= V = xyz$

Surface Area: $12 = 2xy + 2yz + xz$

$$12 - xz = 2xy + 2yz$$

$$12 - xz = y(2x + 2z)$$

$$\frac{12 - xz}{2x + 2z} = y$$

$$V = xyz = x\left(\frac{12 - xz}{2x + 2z}\right)z = \frac{12xz - x^2z^2}{2x + 2z} = V(x, z)$$

$$V_x(x,z) = \frac{(2x+2z)\frac{\partial}{\partial x}(12xz - x^2z^2) - (12xz - x^2z^2)\frac{\partial}{\partial x}(2x+2z)}{(2x+2z)^2}$$

$$= \frac{(2x+2z)(12z - 2xz^2) - (12xz - x^2z^2)(2)}{(2x+2z)^2}$$

$$= \frac{24\cancel{x}z + 24z^2 - 4x^2z^2 - 4xz^3 - 24\cancel{x}z + 2x^2z^2}{(2x+2z)^2}$$

$$= \frac{24z^2 - 2x^2z^2 - 2xz^3}{2^2(x+z)^2} = \frac{2z^2(12 - x^2 - xz)}{4(x+z)^2}$$

$$V_x(x,z) = \frac{z^2(12 - x^2 - 2xz)}{2(x+z)^2}$$

Because of the symmetry in $V = \frac{12xz - x^2z^2}{2x + 2z}$ (trading roles of variables gives same eqn),

we can use V_x to write V_z:

$$V_z(x,z) = \frac{x^2(12 - z^2 - 2xz)}{2(x+z)^2}$$

Set $V_x = V_z = 0$: (so numerators must be 0)

$$z^2(12 - x^2 - 2xz) = 0$$

$$x^2(12 - z^2 - 2xz) = 0$$

Note: if $x = 0, y = 0$, or $z = 0$, we don't have a box. so, throw out $x^2 = 0$, $z^2 = 0$

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$12 - x^2 - 2xz = 0$

$12 - z^2 - 2xz = 0$

Subtract eqns: $0 - x^2 + z^2 + 0 = 0$

$x^2 = z^2$

$x = z$ (can't have negative number for the dimension of a box)

$12 - x^2 - 2xz = 0$

$x = z \Rightarrow$ $12 - x^2 - 2x(x) = 0$

$12 - x^2 - 2x^2 = 0$

$12 - 3x^2 = 0$

$12 = 3x^2$

$4 = x^2$

$2 = x$

$z = 2$ also because $x = z$

Find y: $\dfrac{12 - xz}{2x + 2z} = y$ (from earlier)

$x = 2,\ z = 2 \Rightarrow$ $y = \dfrac{12 - 2(2)}{2(2) + 2(2)} = \dfrac{12-4}{8} = \dfrac{8}{8} = 1$

Maximum Volume is $2m(2m)(1m) = \boxed{4m^3 \text{ Maximum Volume}}$

It occurs when the base of the box is $2m \times 2m$, and the height is $1m$.

Example 5: The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.