14.1: Iterated Integrals and Area in the Plane

Suppose f is a function of two variables that is continuous on the rectangle $R = [a,b] \times [c,d]$.

The notation $\int_c^d f(x,y)\,dy$ means that we consider x to be fixed (constant), and we integrate $f(x,y)$ with respect to y from $y = c$ to $y = d$. This is called *partial integration*. The result is a function of x:

$$A(x) = \int_c^d f(x,y)\,dy.$$

This new function $A(x)$ can be integrated with respect to x from $x = a$ to $x = b$, resulting in:

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

We omit the brackets and write $\int_a^b \int_c^d f(x,y)\,dy\,dx$, called an *iterated integral*.

Note: The order of integration is "from the inside out."

Similarly, the notation $\int_a^b f(x,y)\,dx$ means that we consider y to be fixed (constant), and we integrate $f(x,y)$ with respect to x from $x = a$ to $x = b$. The result is a function of y:

$$B(y) = \int_a^b f(x,y)\,dx.$$

This new function $B(y)$ can be integrated with respect to y from $y = c$ to $y = d$, resulting in:

$$\int_c^d B(y)\,dx = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Example 1: Calculate $\int_x^{x^2} \frac{y}{x}\,dy$.

Example 2: Calculate $\int_y^{y^2} \frac{y}{x}\,dx$.

Example 3: Calculate $\int_0^2 \int_1^3 2x^2 y^3 \, dy \, dx$ and $\int_1^3 \int_0^2 2x^2 y^3 \, dx \, dy$.

> Definition: (Double Integral) (See Section 14.2 in Larson book.)
>
> Suppose $f(x, y)$ is defined on a closed, bounded region *R* in the plane. Also suppose that *R* is partitioned into *n* rectangles in such a way that the norm of the partition (diagonal of the largest rectangle, denoted $\|\Delta\|$) approaches 0 as the number of rectangles approaches infinity. (In other words, $\|\Delta\| \to 0$ as $n \to \infty$). Then the double integral of *f* over *R* is
>
> $$\iint_R f(x, y) \, dA = \lim_{\|\Delta\| \to 0, n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \, \Delta A_i .$$
>
> where ΔA_i is the area of the *i*th rectangle, and (x_i, y_i) is any point in the *i*th rectangle (provided this limit exists).

Using double integrals to find area:

To find area of a region, we integrate the constant function $f(x, y) = 1$. (Because if $f(x, y) = 1$, then $f(x_i, y_i) \, \Delta A_i = \Delta A_i$. If we add up all the areas ΔA_i, we can approximate the area of our region.)

We'll also need the following theorem, which allows us to break down our double integral $\iint_R f(x, y) \, dA$ into an iterated integral using dx and dy.

> Fubini's Theorem: (Minor League Version)
>
> If $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$, then
>
> $$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy .$$

Example 4: Use an iterated integral to find the area of the region described by the inequalities $1 \le x \le 5$, $2 \le y \le 7$.

Theorem: (Area of a Region)

The area of the region bounded by the graphs of $y = g_1(x)$, $y = g_2(x)$, $x = a$, $x = b$ is given by

$$\int_a^b \int_{g_1(x)}^{g_1(x)} dy\,dx,$$ provided g_1 and g_2 are continuous..

The area of the region bounded by the graphs of $x = h_1(y)$, $x = h_2(y)$, $y = c$, $y = d$ is given by

$$\int_c^d \int_{h_1(y)}^{h_1(y)} dx\,dy,$$ provided h_1 and h_2 are continuous.

Example 5: Find the area of the triangle bounded by the graphs of $y = 2x$, $y = 0$, and $x = 3$.

Example 6: Set up integrals to find the area of the region bounded by the graphs of $y = \sqrt{x}$ and $x = y^2$.

Example 7: Find the area of the region bounded by the graphs of $y = 2x$ and $y = x^{3/2}$.

Example 8: Find the area of the region bounded by the graphs of $xy = 9$, $y = x$ and $y = 0$, and $x = 9$.

Switching the order of integration:

Example 9: For the given iterated integral, sketch the region of integration and then switch the order of integration.

$$\int_0^4 \int_{\sqrt{y}}^2 f(x, y)\, dx\, dy$$

Example 10: For the given iterated integral, sketch the region of integration and then switch the order of integration.

$$\int_0^4 \int_0^{4-x^2} f(x, y)\, dy\, dx$$

Example 11: Sketch the region R whose area is given by the iterated integral. Switch the order of integration and show that both orders yield the same area.

$$\int_{-2}^2 \int_0^{4-y^2} dx\, dy$$

Example 12: Sketch the region R whose area is given by the iterated integral. Switch the order of integration and show that both orders yield the same area.

$$\int_0^4 \int_0^{x/2} dy\, dx + \int_4^6 \int_0^{6-x} dy\, dx$$

Example 13: Calculate the iterated integral.

$$\int_0^2 \int_x^2 e^{-y^2}\, dy\, dx$$

Example 14: Calculate the iterated integral.

$$\int_1^2 \int_0^\pi x\cos(xy)\,dx\,dy$$