14.1: Iterated Integrals and Area in the Plane

Suppose *f* is a function of two variables that is continuous on the rectangle $R = [a,b] \times [c,d]$.

The notation $\int_c^d f(x,y)\,dy$ means that we consider *x* to be fixed (constant), and we integrate $f(x,y)$ with respect to y from $y = c$ to $y = d$. This is called *partial integration*. The result is a function of *x*:

$$A(x) = \int_c^d f(x,y)\,dy.$$

This new function $A(x)$ can be integrated with respect to *x* from $x = a$ to $x = b$, resulting in:

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

We omit the brackets and write $\int_a^b \int_c^d f(x,y)\,dy\,dx$, called an *iterated integral*.

Note: The order of integration is "from the inside out."

Similarly, the notation $\int_a^b f(x,y)\,dx$ means that we consider *y* to be fixed (constant), and we integrate $f(x,y)$ with respect to *x* from $x = a$ to $x = b$. The result is a function of *y*:

$$B(y) = \int_a^b f(x,y)\,dx.$$

This new function $B(y)$ can be integrated with respect to *y* from $y = c$ to $y = d$, resulting in:

$$\int_c^d B(y)\,\cancel{dx}\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Ex $1\frac{1}{2}$: $\int_1^2 \frac{y}{x}\,dy$

$= \frac{1}{x}\int_1^2 y\,dy$

$= \frac{1}{x}\cdot\frac{y^2}{2}\Big|_1^2$

$\frac{2^2}{2x} - \frac{1^2}{2x} = \boxed{\frac{3}{2x}}$

Example 1: Calculate $\int_x^{x^2} \frac{y}{x}\,dy$.

$$\int_x^{x^2} \frac{y}{x}\,dy = \frac{1}{x}\int_x^{x^2} y\,dy = \frac{1}{x}\cdot\frac{y^2}{2}\Big|_x^{x^2}$$

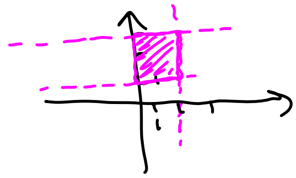
$$= \frac{(x^2)^2}{2x} - \frac{x^2}{2x} = \frac{x^4 - x^2}{2x} = \boxed{\frac{x^3 - x}{2}}$$

Example 2: Calculate $\int_y^{y^2} \frac{y}{x}\,dx$.

$$\int_y^{y^2} y\left(\frac{1}{x}\right)dx = y\ln|x|\Big[_y^{y^2} = y\ln|y^2| - y\ln|y|$$

$$= y\ln y^2 - y\ln|y| = y\ln|y|^2 - y\ln|y|$$

$$= 2y\ln|y| - y\ln|y| = \boxed{y\ln|y|}$$

Region of integration: 

Example 3: Calculate $\int_0^2 \int_1^3 2x^2y^3\ dy\ dx$ and $\int_1^3 \int_0^2 2x^2y^3\ dx\ dy$.

$$\int_0^2\int_1^3 2x^2y^3\,dy\,dx = \int_0^2 \frac{2x^2y^4}{4}\Big|_1^3 dx$$

$$= \int_0^2 \left[\frac{2x^2(3)^4}{4} - \frac{2x^2(1)^4}{4}\right]dx$$

$$= \int_0^2\left[\frac{81x^2}{2} - \frac{x^2}{2}\right]dx = \int_0^2 \frac{80x^2}{2}\,dx = \int_0^2 40x^2\,dx$$

$$= \frac{40x^3}{3}\Big|_0^2 = \frac{40(2)^3}{3} - \cancel{\frac{40(0)^3}{3}}^{\,0} = \boxed{\frac{320}{3}}$$

$$\int_1^3\int_0^2 2x^2y^3\,dx\,dy = \int_1^3 \frac{2x^3y^3}{3}\Big|_0^2 dy$$

$$= \int_1^3\left[\frac{2(2)^3y^3}{3} - \cancel{\frac{2(0)^3y^3}{3}}^{\,0}\right]dy$$

$$= \int_1^3 \frac{16}{3}y^3\,dy = \frac{16}{3}\cdot\frac{y^4}{4}\Big|_1^3$$

$$\frac{4}{3}y^4\Big|_1^3 = \frac{4}{3}(3)^4 - \frac{4}{3}(1)^4 = \frac{4(81)}{3} - \frac{4}{3} = \frac{324}{3} - \frac{4}{3} = \boxed{\frac{320}{3}}$$

equal!

<u>Definition</u>: (Double Integral) (See Section 14.2 in Larson book.)

Suppose $f(x,y)$ is defined on a closed, bounded region R in the plane. Also suppose that R is partitioned into n rectangles in such a way that the norm of the partition (diagonal of the largest rectangle, denoted $\|\Delta\|$) approaches 0 as the number of rectangles approaches infinity. (In other words, $\|\Delta\| \to 0$ as $n \to \infty$). Then the double integral of f over R is

$$\iint_R f(x,y)\,dA = \lim_{\|\Delta\|\to 0,\, n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i.$$

where ΔA_i is the area of the ith rectangle, and (x_i, y_i) is any point in the ith rectangle (provided this limit exists).

Using double integrals to find area:

To find area of a region, we integrate the constant function $f(x,y) = 1$. (Because if $f(x,y) = 1$, then $f(x_i, y_i)\,\Delta A_i = \Delta A_i$. If we add up all the areas ΔA_i, we can approximate the area of our region.)

We'll also need the following theorem, which allows us to break down our double integral $\iint_R f(x,y)\,dA$ into an iterated integral using dx and dy.

<u>Fubini's Theorem</u>: (Minor League Version)

If $f(x,y)$ is continuous on the rectangle $R = [a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

$\{(x,y) \mid x\in[a,b] \text{ and } y\in[c,d]\}$

$a \le x \le b$

$c \le y \le d$

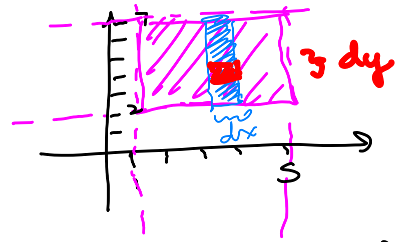
Geometry: width $= 5-1=4$, height $= 7-2=5$ $\Big\}$ Area $= 5(4) = 20$ ✓

Example 4: Use an iterated integral to find the area of the region described by the inequalities $1 \le x \le 5$, $2 \le y \le 7$.

$$\text{Area} = \int_1^5 \underbrace{\int_2^7 1\,dy}_{\text{height of blue rectangle}}\,dx = \int_1^5 y\Big|_2^7\,dx$$

$$= \int_1^5 (7-2)\,dx$$

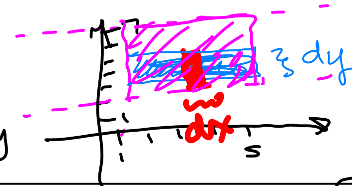
$$= \int_1^5 5\,dx = 5x\Big|_1^5 = 5(5)-5(1) = 25-5 = \boxed{20}$$



$$\text{Area} = \int_2^7 \int_1^5 1\,dx\,dy$$

$$= \int_2^7 x\Big|_1^5\,dy = \int_2^7 (5-1)\,dy = \int_2^7 4\,dy$$

$$= 4y\Big|_2^7 = 4(7)-4(2) = 28-8 = \boxed{20}$$



Theorem: (Area of a Region)

The area of the region bounded by the graphs of $y = g_1(x)$, $y = g_2(x)$, $x = a$, $x = b$ is given by

$\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$

$$\int_a^b \int_{g_1(x)}^{g_1(x)} dy\,dx, \qquad \text{provided } g_1 \text{ and } g_2 \text{ are continuous..}$$

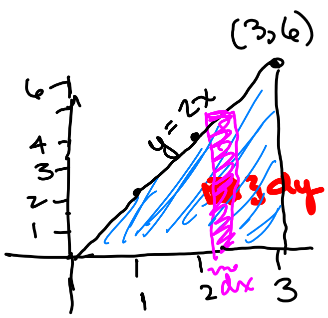
The area of the region bounded by the graphs of $x = h_1(y)$, $x = h_2(y)$, $y = c$, $y = d$ is given by

$\int_c^d \int_{h_1(y)}^{h_2(y)} dx\,dy$

$$\int_c^d \int_{h_1(y)}^{h_1(y)} dx\,dy, \qquad \text{provided } h_1 \text{ and } h_2 \text{ are continuous.}$$

Example 5: Find the area of the triangle bounded by the graphs of $y = 2x$, $y = 0$, and $x = 3$.

Geometry: $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(3)(6) = \frac{18}{2} = 9$ ✓



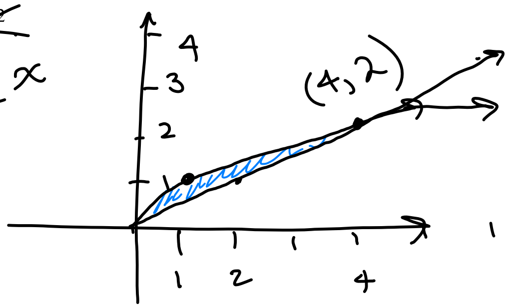
$$\text{Area} = \int_0^3 \int_0^{y=2x} 1\,dy\,dx$$

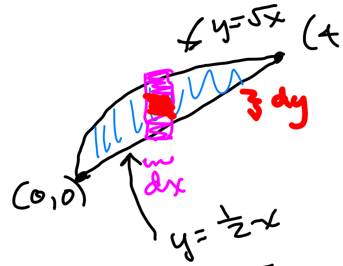
$$= \int_0^3 y\Big|_{y=0}^{2x}\,dx = \int_0^3 (2x-0)\,dx$$

$$= \int_0^3 2x\,dx = \frac{2x^2}{2}\Big|_0^3 = x^2\Big|_0^3 = 3^2-0^2 = \boxed{9}$$

Example 6: Set up integrals to find the area of the region bounded by the graphs of $y = \sqrt{x}$ and ~~$x = y^2$~~

$y = \frac{1}{2}x$





$$\text{Area} = \int_0^4 \int_{\frac{1}{2}x}^{\sqrt{x}} 1\,dy\,dx$$

$$\left.\begin{array}{l} y = \frac{1}{2}x \\ y = \sqrt{x} \end{array}\right\}$$

$\frac{1}{2}x = \sqrt{x}$

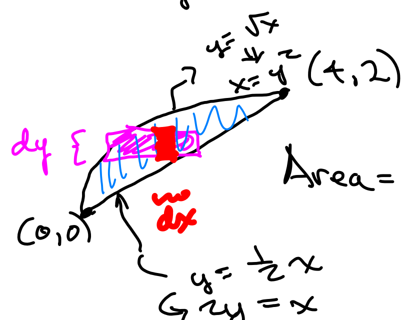
$(\frac{1}{2}x)^2 = (\sqrt{x})^2$

$\frac{x^2}{4} = x$

$x^2 = 4x \Rightarrow x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, 4$

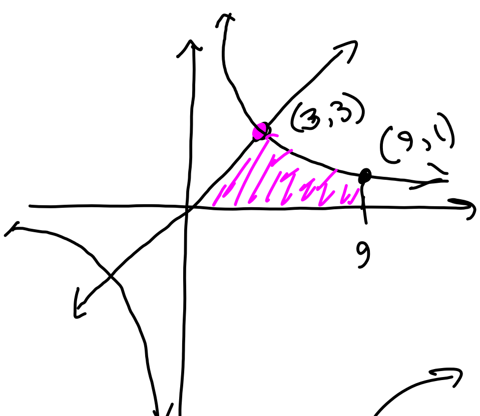


$$\text{Area} = \int_0^2 \int_{y^2}^{2y} 1\,dx\,dy$$

Example 7: Find the area of the region bounded by the graphs of $y = 2x$ and $y = x^{3/2}$.

Example 8: Find the area of the region bounded by the graphs of $xy = 9$, $y = x$ and $y = 0$, and $x = 9$.

$xy = 9 \Rightarrow y = \frac{9}{x} = 9\left(\frac{1}{x}\right)$



Intersect $xy = 9$ with $y = x$:

$x(x) = 9$

$x^2 = 9$

$x = \pm 3$

$y = 3$

Intersect $x = 9$ with $xy = 9$

$9y = 9$

$y = 1$

$$\text{Area} = A_1 + A_2 = \int_0^3 \int_0^x 1\,dy\,dx + \int_3^9 \int_0^{9/x} 1\,dy\,dx$$

Switching the order of integration:

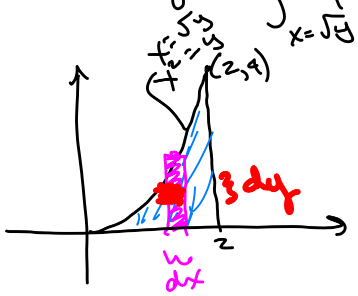
Example 9: For the given iterated integral, sketch the region of integration and then switch the order of integration.

$$\int_0^4\int_{\sqrt{y}}^2 f(x,y)\,dx\,dy$$

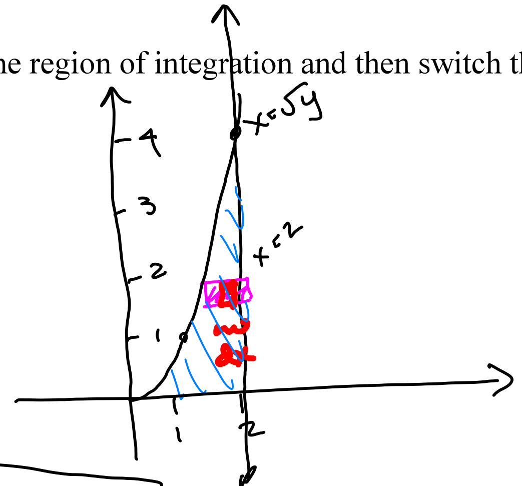
$y = x^2$ $(x \ge 0)$

dx runs from $x=\sqrt{y}$ to $x=2$

Inner integral $\int_{x=\sqrt{y}}^{x=2} f(x,y)\,dx$



$$\boxed{\int_0^2\int_0^{x^2} f(x,y)\,dy\,dx}$$



Example 10: For the given iterated integral, sketch the region of integration and then switch the order of integration.

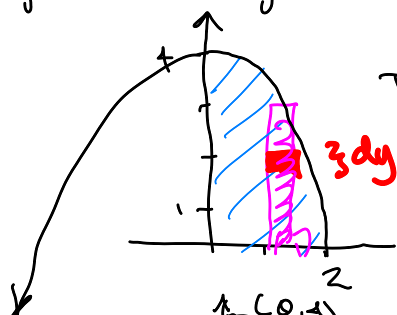
Inside integral: dy goes from $y=0$ to $y=4-x^2$

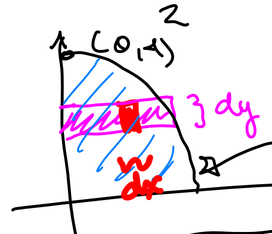
$y=-x^2+4$

$$\int_0^{\cancel{x}\,2}\int_0^{4-x^2} f(x,y)\,dy\,dx$$

$$\boxed{\int_0^4\int_0^{\sqrt{4-y}} f(x,y)\,dx\,dy}$$

Then dx runs from $x=0$ to $x=4$





$y=4-x^2$

$y-4=-x^2$

$4-y=x^2$

$x=\pm\sqrt{4-y}$

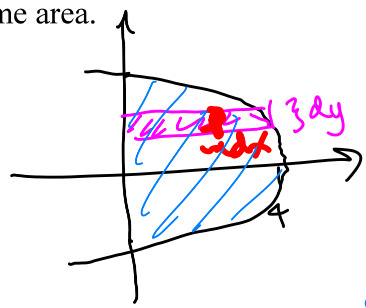
I want $x=+\sqrt{4-y}$

Example 11: Sketch the region *R* whose area is given by the iterated integral. Switch the order of integration and show that both orders yield the same area.

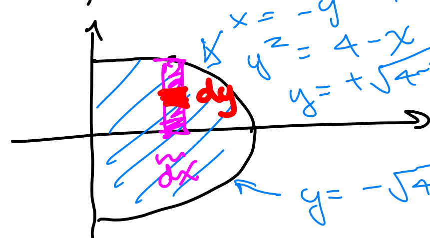
$$\int_{-2}^2\int_0^{4-y^2} dx\,dy$$

dx runs from $x=0$ to $x=4-y^2$

$x=-y^2+4$



$$\text{Area} = 2\int_0^4\int_0^{\sqrt{4-x}} 1\,dy\,dx$$



$x=-y^2+4$

$y^2=4-x$

$y=+\sqrt{4-x}$

$y=-\sqrt{4-x}$

Example 12: Sketch the region *R* whose area is given by the iterated integral. Switch the order of integration and show that both orders yield the same area.

$$\underbrace{\int_0^4\int_0^{x/2 \;(\frac{1}{2}x)} dy\,dx}_{I_1} + \underbrace{\int_4^6\int_0^{6-x} dy\,dx}_{I_2}$$

$y = \frac{1}{2}x$

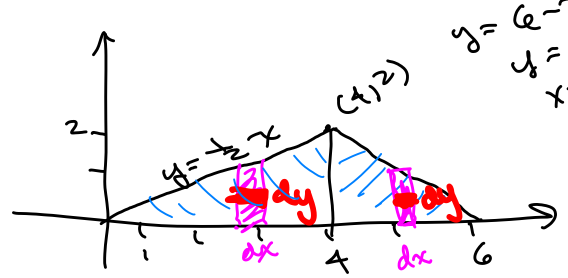
$y = 6-x$

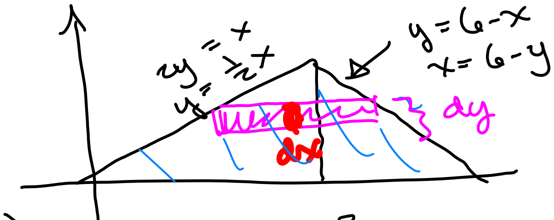
$y = -x+6$

$x=6 \Rightarrow y = -6+6 = 0$

$x=4 \Rightarrow y = -4+6 = 2$

$(4,2)$





$$\text{Area} = \int_0^2\int_{2y}^{6-y} 1\,dx\,dy$$

$$= \int_0^2 x\Big|_{2y}^{6-y}\,dy$$

$$= \int_0^2 (6-y-2y)\,dy = \int_0^2 (6-3y)\,dy = 6y - \frac{3y^2}{2}\Big|_0^2 = 6(2) - \frac{3(2)^2}{2} - 0$$

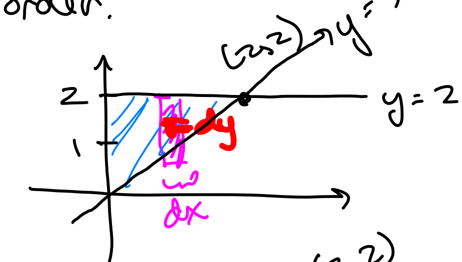
$$= 12 - 6 = \boxed{6}$$

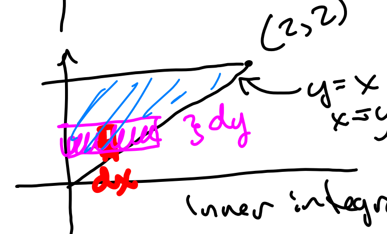
Example 13: Calculate the iterated integral.

$$\int_0^2\int_x^2 e^{-y^2}\,dy\,dx$$

You have no choice except to switch the order.

Inner integral: dy runs from $y=x$ to $y=2$





$$\int_0^2\int_0^y e^{-y^2}\,dx\,dy$$

Inner integral: dx runs from $x=0$ to $x=$

$$= \int_0^2 xe^{-y^2}\Big|_0^y\,dy = \int_0^2 (ye^{-y^2} - 0e^{-y^2})\,dy$$

$$= \int_0^2 \underbrace{y\,\underbrace{e^{-y^2}}_{e^u}\,dy}_{-\frac{1}{2}du} = -\frac{1}{2}\int_{y=0}^{y=2} e^u\,du = -\frac{1}{2}e^u\Big|_{y=0}^{y=2}$$

$u = -y^2$

$\frac{du}{dy} = -2y$

$du = -2y\,dy$

$-\frac{1}{2}du = y\,dy$

$$= -\frac{1}{2}e^{-y^2}\Big|_0^2 = -\frac{1}{2}e^{-2^2} + \frac{1}{2}e^{-0^2}$$

$$= -\frac{1}{2}e^{-4} + \frac{1}{2}(1) = -\frac{1}{2e^4} + \frac{1}{2}$$

$$= \frac{-1}{2e^4} + \frac{e^4}{2e^4} = \boxed{\frac{e^4-1}{2e^4}}$$

Example 14: Calculate the iterated integral.

$$\int_1^2 \int_0^{\pi} x\cos(xy)\, dx\, dy$$