14.2: Double Integrals and Volume

Definition: (Double Integral)

Suppose $f(x, y)$ is defined on a closed, bounded region *R* in the plane. Also suppose that *R* is partitioned into *n* rectangles in such a way that the norm of the partition (diagonal of the largest rectangle, denoted $\|\Delta\|$) approaches 0 as the number of rectangles approaches infinity. (In other words, $\|\Delta\| \to 0$ as $n \to \infty$). Then the double integral of *f* over *R* is

$$\iint_R f(x, y)\, dA = \lim_{\|\Delta\| \to 0, n \to \infty} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i .$$

where ΔA_i is the area of the *i*th rectangle, and (x_i, y_i) is any point in the *i*th rectangle (provided this limit exists). If this limit exists, then *f* is *integrable* over *R*.

Volume of a Solid Region

If *f* is integrable over a plane region *R* and $f(x, y) \geq 0$ for all (x, y) in *R*, then the volume of the solid region that lies above *R* and below the graph of *f* is

$$\iint_R f(x, y)\, dA .$$

Properties of Double Integrals

1. $\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA$

2. $\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$

3. $\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$

4. $\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$

5. $\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$, where R is the union of two nonoverlapping subregions R_1 and R_2.

To evaluate a double integral $\iint\limits_R f(x, y)\, dA$, we must rewrite it as an iterated integral. (We replace dA by either $dy\,dx$ or by $dx\,dy$, and we replace R by the corresponding limits of integration.

Fubini's Theorem:

Suppose $f(x, y)$ is continuous on the plane region R.

If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a,b]$, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx .$$

R is defined by $c \le y \le d$ and $g_1(x) \le y \le g_2(x)$, where h_1 and h_2 are continuous on $[c,d]$, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy .$$

Example 1: Evaluate $\iint_R \frac{y}{1+x^2}\, dA$, where R is the region bounded by the graphs of $y=0$, $y=\sqrt{x}$, and $x=4$.

Example 2: Evaluate $\iint_R \sin x \sin y \, dA$, where R is the rectangle with vertices $(-\pi,0)$, $(\pi,0)$, $(\pi,\frac{\pi}{2})$, and $(-\pi,\frac{\pi}{2})$.

Example 3: Evaluate $\iint_R (2x+3y+7)\, dA$, where R is the region bounded by the graphs of $y = \frac{1}{3}x$ and $y = \sqrt{x}$.

Example 4: Find the volume of the solid bounded above by the surface $x+2y+3z=6$ in the first octant.

Example 5: Find the volume of the solid in the first octant bounded by the graphs of $y = 4 - x^2$ and $z = 4 - x^2$.

Average value of a function:

<u>Definition</u>: <u>Average Value of a Function</u>

If f is integrable over a plane region R, then the average value of f over R is

$$\text{Average value} = \frac{1}{A}\iint_R f(x, y)\, dA,$$

where A is the area of R.

Example 6: Find the average value of $f(x, y) = \sin(x + y)$ on the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,\frac{\pi}{2})$ and $(0,\frac{\pi}{2})$.