14.3: Change of Variables: Polar Coordinates

For some regions, using polar coordinates makes sense (and makes the integration easier)!

Recall: To convert between polar and rectangular coordinates, use the following equations:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x}$$

For regions that are variations of circles, we can write dA in terms of r and θ:

Theorem: Changing a Double Integral to Polar Form

Let R be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Example 1: Use a double integral to calculate the area of a circle of radius 3.

Example 2: For $\int_0^{\pi/4}\int_0^4 r^2 \cos\theta r \sin\theta \, dr \, d\theta$, sketch the region of integration and calculate the iterated integral.

Example 3: Calculate $\iint_R (x+y)\, dA$, where R is the disk bounded by $x^2+y^2=25$.

Example 4: Evaluate $\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\, dx\, dy$ by converting to polar coordinates.

Example 5: Calculate $\iint_R f(x,y)\, dA$, where R is the region described by $x^2+y^2 \le 25$, $x \ge 0$, and $f(x,y) = e^{-(x^2+y^2)/2}$.

Example 6: Calculate $\iint_R f(x,y)\,dA$, where R is the region described by $x^2 + y^2 \le 9$, $x \ge 0$, $y \ge 0$ and $f(x,y) = 9 - x^2 - y^2$.

Example 7: Use a double integral to find the area enclosed by the graph of $r = 3\cos 3\theta$.

Example 8: Find the volume of the solid bounded by the paraboloid $z = 10 - 3x^2 - 3y^2$ and the plane $z = 4$.

Example 9: Find the volume of the solid inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = 1$.

Example 10: Find the area of the region inside the graph of $r = 2\cos\theta$ and outside the graph of $r = 1$.

Example 11: Find the area of the region enclosed by the graph of $r = 2 + \sin\theta$.

Example 12: Find the volume of the sphere of radius R.