14.3: Change of Variables: Polar Coordinates

For some regions, using polar coordinates makes sense (and makes the integration easier)!

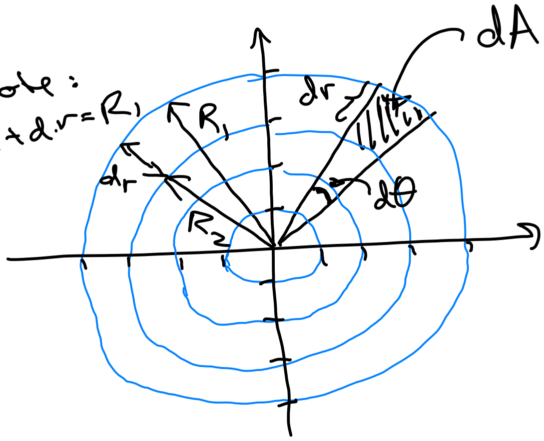
Recall: To convert between polar and rectangular coordinates, use the following equations:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x}$$

For regions that are variations of circles, we can write dA in terms of r and θ:

Note: $R_2 + dr = R_1$



Polar rectangle:

$\{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$

How to find dA?

Area of a sector cut off by central angle θ?

$$\frac{A_{sector}}{\pi r^2} = \frac{\theta}{2\pi} \Rightarrow A_{sector} = \frac{\pi r^2 \theta}{2\pi} = \frac{r^2\theta}{2} = \frac{1}{2} r^2 \theta$$

Here, $dA = \text{Area}_{\text{large sector}} - \text{Area}_{\text{small sector}}$

$$= \frac{1}{2}(R_1)^2\,d\theta - \frac{1}{2}(R_2)^2\,d\theta$$

$$= \frac{1}{2}(R_2 + dr)^2\,d\theta - \frac{1}{2}R_2^2\,d\theta$$

$$= \frac{1}{2}(\cancel{R_2^2} + 2R_2\,dr + (dr)^2 - \cancel{R_2^2})\,d\theta = \frac{2R_2\,dr + (dr)^2}{2}\,d\theta$$

$$\frac{dr\,(2R_2 + dr)}{2}\,d\theta = dr(r)\,d\theta = r\,dr\,d\theta$$

Let r = average of R_1 and R_2

$$r = \frac{R_1 + R_2}{2} \text{ (avg. radius)}$$

$$= \frac{R_2 + dr + R_2}{2} = \frac{2R_2 + dr}{2}$$

Theorem: Changing a Double Integral to Polar Form

Let R be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

Check using geometry formula

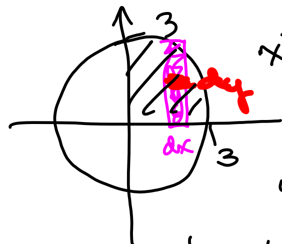
$A = \pi r^2 = \pi (3)^2 = 9\pi \checkmark$

Example 1: Use a double integral to calculate the area of a circle of radius 3.

$$\text{Area} = 4\int_0^3 \int_0^{\sqrt{9-x^2}} dy\,dx$$

$$= 4\int_0^3 y\Big|_0^{\sqrt{9-x^2}} dx = 4\int_0^3 \sqrt{9-x^2}\,dx$$

use trig substitution to integrate



$x^2+y^2=9$

$y^2 = 9-x^2$

$y = \pm\sqrt{9-x^2}$

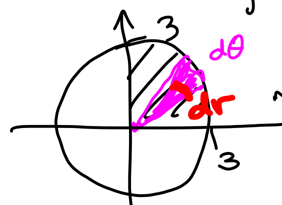
top half: $y=\sqrt{9-x^2}$

In polar,

$$\text{Area} = 4\int_0^{\pi/2}\int_0^3 r\,dr\,d\theta = 4\int_0^{\pi/2} \frac{r^2}{2}\Big|_0^3 d\theta$$

$$= 4\int_0^{\pi/2}\left[\frac{3^2}{2} - \frac{0^2}{2}\right] d\theta = 4\int_0^{\pi/2} \frac{9}{2}\,d\theta$$

$$= \frac{36}{2}\,\theta\Big|_0^{\pi/2} = 18\left(\frac{\pi}{2}-0\right) = \boxed{9\pi}$$



$x^2+y^2=9$

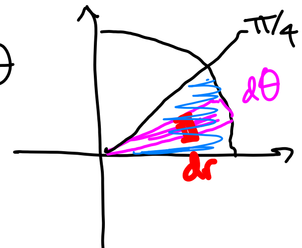
Example 2: For $\int_0^{\pi/4}\int_0^4 r^2\cos\theta\sin\theta\,dr\,d\theta$, sketch the region of integration and calculate the iterated integral.

$$\int_0^{\pi/4}\int_0^4 r^2\cos\theta\sin\theta\,dr\,d\theta = \int_0^{\pi/4} \frac{r^3}{3}\cos\theta\sin\theta\Big|_0^4 d\theta$$

$$= \int_0^{\pi/4} \frac{4^3}{3}\cos\theta\sin\theta\,d\theta = \frac{64}{3}\int_0^{\pi/3}\cos\theta\sin\theta\,d\theta$$

$$= \frac{64}{3}\int_0^{\sqrt{2}/2} u\,du = \frac{64}{3}\cdot\frac{u^2}{2}\Big|_0^{\sqrt{2}/2 = \frac{1}{\sqrt{2}}}$$

$$= \frac{64}{6}\left[\left(\frac{1}{\sqrt{2}}\right)^2 - 0^2\right] = \frac{32}{3}\left(\frac{1}{2}\right) = \frac{32}{6} = \boxed{\frac{16}{3}}$$



Sector of a circle

$\theta = 0 \Rightarrow u = \sin 0 = 0$

$\theta = \pi/4 \Rightarrow u = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$du = \cos\theta\,d\theta$

Example 3: Calculate $\iint_R (x+y)\,dA$, where R is the disk bounded by $x^2+y^2=25$.

$$\iint_R (x+y)\,dA = \int_0^{2\pi}\int_0^5 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

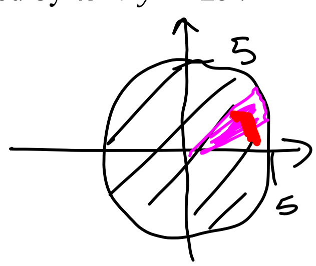
$$= \int_0^{2\pi}\int_0^5 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}(\cos\theta+\sin\theta)\Big|_0^5 d\theta$$

$$= \int_0^{2\pi}\left(\frac{5^3}{3} - \frac{0^3}{3}\right)(\cos\theta+\sin\theta)\,d\theta$$

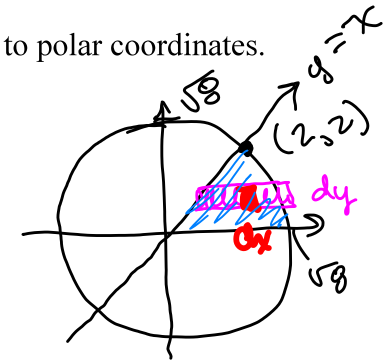
$$= \frac{125}{3}\int_0^{2\pi}(\cos\theta+\sin\theta)\,d\theta = \frac{125}{3}\left[\sin\theta - \cos\theta\right]_0^{2\pi} = \frac{125}{3}\left[\sin 2\pi - \cos 2\pi - \sin 0 + \cos 0\right]$$

$$= \frac{125}{3}\left[0 - (1) - 0 + 1\right] = \boxed{0}$$



$dA = r\,dr\,d\theta$

Example 4: Evaluate $\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\, dx\, dy$ by converting to polar coordinates.



$$\int_0^2 \int_y^{\sqrt{8-y^2}} \underbrace{\sqrt{x^2+y^2}}_{r}\, dx\, dy$$

$$= \int_0^{\pi/4} \int_0^{\sqrt{8}} (r)\, r\, dr\, d\theta \Leftarrow \begin{cases} x^2+y^2=r^2, \\ \text{so } r=\sqrt{x^2+y^2} \end{cases}$$

$$= \int_0^{\pi/4} \int_0^{\sqrt{8}} r^2\, dr\, d\theta$$

$$= \int_0^{\pi/4} \left.\frac{r^3}{3}\right|_0^{\sqrt{8}} d\theta = \int_0^{\pi/4} \left(\frac{(\sqrt{8})^3}{3} - 0\right) d\theta$$

$$= \int_0^{\pi/4} \frac{\sqrt{8}\sqrt{8}\sqrt{8}}{3}\, d\theta = \frac{8\sqrt{8}}{3} \int_0^{\pi/4} d\theta$$

$$= \frac{8\sqrt{8}}{3} \cdot \theta \Big|_0^{\pi/4} = \frac{8 \cdot 2\sqrt{2}}{3}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{16\sqrt{2}\pi}{12} = \boxed{\frac{4\sqrt{2}\pi}{3}}$$

dx runs from $x=y$ to $x=\sqrt{8-y^2}$

$x^2 = 8-y^2$

$x^2+y^2 = 8$

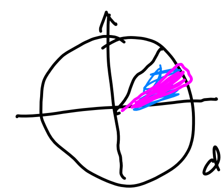
Circle of radius $\sqrt{8}$

Where do circle and line intersect? $x^2+y^2=8$

$y=x \Rightarrow x^2+x^2=8$

$2x^2=8$

$x^2=4 \Rightarrow x=\pm 2$

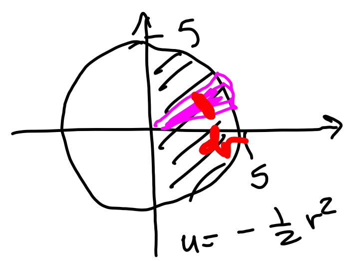


$r\,dr$ goes from 0 to $\sqrt{8}$

$d\theta$ goes from 0 to $\frac{\pi}{4}$

Example 5: Calculate $\iint_R f(x,y)\, dA$, where R is the region described by $x^2+y^2 \le 25$, $x \ge 0$, and $f(x,y) = e^{-(x^2+y^2)/2}$.

$f(x,y) = e^{-\frac{x^2+y^2}{2}}$; $x^2+y^2=r^2$



$$\iint_R f(x,y)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^5 e^{-\frac{r^2}{2}}\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} (-1) \int_0^{-25/2} e^u\, du\, d\theta = -\int_{-\pi/2}^{\pi/2} e^u \Big|_0^{-25/2} d\theta$$

$u = -\frac{1}{2}r^2$

$\frac{du}{dr} = -\frac{1}{2}(2r) = -r$

$-du = r\,dr$

$r=0 \Rightarrow u = -\frac{1}{2}(0)^2 = 0$

$r=5 \Rightarrow u = -\frac{1}{2}(5)^2 = -25/2$

$$= -\int_{-\pi/2}^{\pi/2} \left(e^{-25/2} - e^0\right) d\theta$$

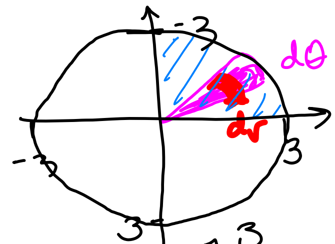
$$= \int_{-\pi/2}^{\pi/2} \left(1 - e^{-25/2}\right) d\theta$$

$$= \left[\theta - \theta e^{-25/2}\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2} - \frac{\pi}{2}e^{-25/2} - \left[-\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)e^{-\frac{25}{2}}\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{2}e^{-25/2} + \pi/2 - \frac{\pi}{2}e^{-25/2} = \boxed{\pi - \pi e^{-25/2}}$$

Example 6: Calculate $\iint_R f(x,y)\,dA$, where R is the region described by $x^2+y^2 \le 9$, $x \ge 0$, $y \ge 0$ and $f(x,y) = 9 - x^2 - y^2$.

$$9-x^2-y^2 = 9-(x^2+y^2) = 9-r^2$$



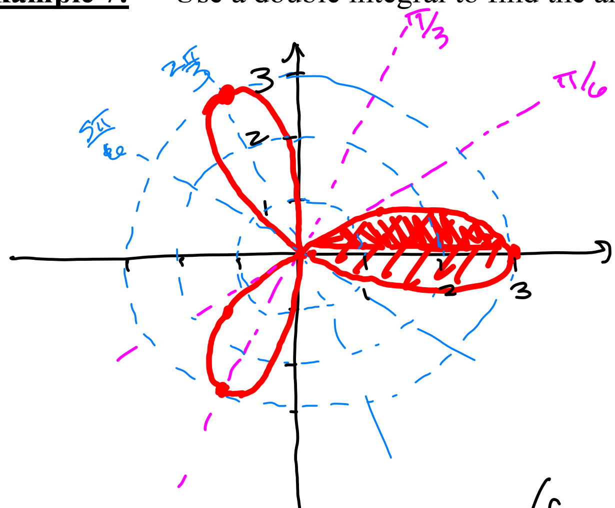
$$\iint_R (9-x^2-y^2)\,dA = \int_0^{\pi/2}\int_0^3 (9-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^3 (9r - r^3)\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{9r^2}{2} - \frac{r^4}{4}\right]\Big|_0^3 d\theta$$

$$= \int_0^{\pi/2}\left[\frac{9(3)^2}{2} - \frac{3^4}{4}\right]d\theta = \int_0^{\pi/2}\left[\frac{81}{2} - \frac{81}{4}\right]d\theta = \int_0^{\pi/2}\left[\frac{162}{4} - \frac{81}{4}\right]d\theta$$

$$= \int_0^{\pi/2}\frac{81}{4}\,d\theta = \frac{81}{4}\theta\Big|_0^{\pi/2} = \frac{81}{4}\left(\frac{\pi}{2}\right) - \frac{81}{4}(0) = \boxed{\frac{81\pi}{8}}$$

Example 7: Use a double integral to find the area enclosed by the graph of $r = 3\cos 3\theta$.



θ	$r = 3\cos 3\theta$
0	$3\cos 0 = 3(1) = 3$
$\frac{\pi}{18}$	$3\cos\frac{3\pi}{18} = \frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{12}$	$3\cos\frac{3\pi}{12} = \frac{3\sqrt{3}}{2} \approx 2.12$
$\frac{\pi}{9}$	$3\cos\frac{3\pi}{9} = 3\left(\frac{1}{2}\right) = 1.5$
$\frac{\pi}{6}$	$3\cos\frac{3\pi}{6} = 3(0) = 0$
$\frac{4\pi}{18}$	$3\cos\frac{12\pi}{18} = 3\left(-\frac{1}{2}\right) = -1.5$
$\frac{\pi}{3}$	$3\cos\frac{3\pi}{3} = -3$
$\frac{\pi}{2}$	$3\cos\frac{3\pi}{2} = 3(0) = 0$
$\frac{8\pi}{18}$	$3\cos\frac{24\pi}{18} = 3\cos\frac{4\pi}{3} = 3\left(-\frac{1}{2}\right) = -1.5$
$\frac{2\pi}{3}$	$3\cos\frac{6\pi}{3} = 3\cos 2\pi = 3(1) = 3$
$\frac{5\pi}{6}$	$3\cos\frac{15\pi}{6} = 3\cos\frac{5\pi}{2} = 0$
π	$3\cos 3\pi = 3(-1) = -3$
$-\frac{\pi}{6}$	$3\cos -\frac{3\pi}{6} = 0$
$-\frac{\pi}{18}$	$3\cos\left(-\frac{3\pi}{18}\right) \approx 2.6$

You could either do 1 loop (from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$) and multiply by 3, or half a loop (0 to $\frac{\pi}{6}$) and multiply by 6.

$$\text{Area} = 6\int_0^{\pi/6}\int_0^{3\cos 3\theta} r\,dr\,d\theta$$

$\vdots$

Should get $\frac{9\pi}{4}$ See next page for details (I worked it out after class)

Ex 7 cont'd:

$$\text{Area} = 6\int_0^{\pi/6}\int_0^{3\cos 3\theta} r\,dr\,d\theta = 6\int_0^{\pi/6} \frac{r^2}{2}\Big|_0^{3\cos 3\theta} d\theta$$

$$= 3\int_0^{\pi/6} r^2\Big|_0^{3\cos 3\theta} d\theta = 3\int_0^{\pi/6}\left[(3\cos 3\theta)^2 - 0^2\right]d\theta$$

$$= 3\cdot 9\int_0^{\pi/6} \cos^2 3\theta\, d\theta$$

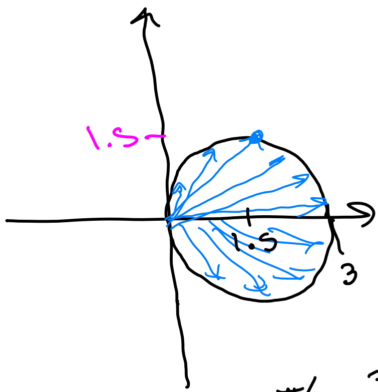
$$= 27\int_0^{\pi/6} \frac{1}{2}\left[1+\cos 6\theta\right]d\theta$$

Half-angle identity
$\cos^2\theta = \frac{1}{2}[1+\cos 2\theta]$
$\cos^2 3\theta = \frac{1}{2}[1+\cos 6\theta]$

$$= \frac{27}{2}\left[\theta + \frac{\sin 6\theta}{6}\right]\Big|_0^{\pi/6} = \frac{27}{2}\left[\frac{\pi}{6} + \frac{1}{6}\sin\left(6\frac{\pi}{6}\right)\right]$$

$$= \frac{27}{2}\left[\frac{\pi}{6} + \frac{1}{6}(0)\right] = \frac{27}{2}\left(\frac{\pi}{6}\right) = \frac{27\pi}{12} = \boxed{\frac{9\pi}{4}}$$

Ex $6\frac{1}{2}$ Use a double integral to find the area enclosed by $r = 3\cos\theta$.



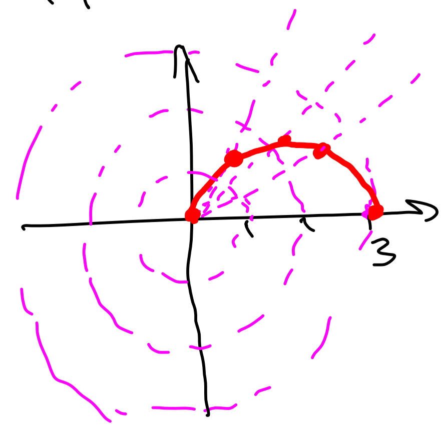
θ	$r = 3\cos\theta$
0	$3\cos 0 = 3(1) = 3$
$\frac{\pi}{3}$	$3\cos\frac{\pi}{3} = 3\left(\frac{1}{2}\right) = 1.5$
$\frac{\pi}{2}$	$3\cos\frac{\pi}{2} = 0$
$-\frac{\pi}{2}$	$3\cos(-\pi/2) = 0$
$\frac{\pi}{6}$	$3\cos\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right) \approx 2.59$
$\frac{\pi}{4}$	$3\cos\frac{\pi}{4} = \frac{3\sqrt{2}}{2} \approx 2.12$

$$\text{Area} = 2\int_0^{\pi/2}\int_0^{3\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \frac{r^2}{2}\Big|_0^{3\cos\theta} d\theta$$

$$= \int_0^{\pi/2}\left[(3\cos\theta)^2 - (0)^2\right] d\theta$$

$$= \int_0^{\pi/2} 9\cos^2\theta\, d\theta$$



$$= 9\int_0^{\pi/2} \frac{1}{2}\left[1+\cos 2\theta\right] d\theta$$

Recall: $\frac{1}{2}$ angle identities

$\cos^2\theta = \frac{1}{2}\left[1 + \cos 2\theta\right]$

$\sin^2\theta = \frac{1}{2}\left[1 - \cos 2\theta\right]$

$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]\Big|_0^{\pi/2}$$

$$= \frac{9}{2}\left[\frac{\pi}{2} + \frac{1}{2}\sin\frac{2\pi}{2} - 0 - \frac{1}{2}\sin 0\right]$$

$$= \frac{9}{2}\left[\frac{\pi}{2} + \frac{1}{2}(0) - 0 - 0\right]$$

$$= \boxed{\frac{9\pi}{4}}$$

Geometry: Area $= \pi r^2$

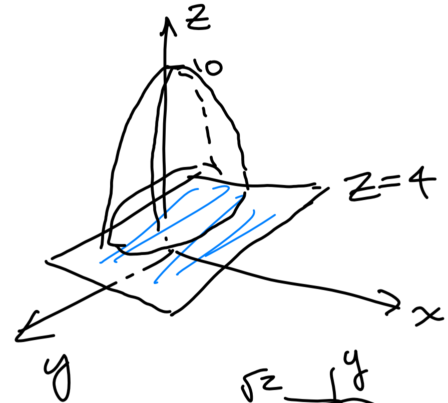
$= \pi(1.5)^2 = \pi\left(\frac{3}{2}\right)^2$

$= \frac{9\pi}{4}$ ✓

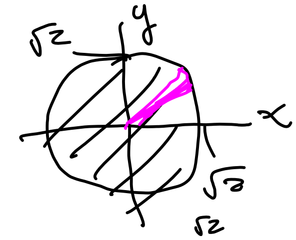
Example 8: Find the volume of the solid bounded by the paraboloid $z = 10 - 3x^2 - 3y^2$ and the plane $z = 4$.

where does the paraboloid intersect the plane? when $z = 4$:

$$4 = 10 - 3x^2 - 3y^2$$
$$3x^2 + 3y^2 = 6$$
$$x^2 + y^2 = 2$$



Modify the function by subtracting 4 from all the z's
paraboloid



$$g(x,y) = z - 4$$
$$= (10 - 3x^2 - 3y^2) - 4$$
$$= 6 - 3x^2 - 3y^2$$
$$g(x,y) = 6 - 3(x^2 + y^2)$$
$$\Rightarrow \hat{g}(r,\theta) = 6 - 3r^2$$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\sqrt{2}} (6 - 3r^2)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} (6r - 3r^3)\, dr\, d\theta$$
$$= \int_0^{2\pi} \left(\frac{6r^2}{2} - \frac{3r^4}{4}\right)\Big|_0^{\sqrt{2}} = \int_0^{2\pi} \left[3(\sqrt{2})^2 - 3\frac{(\sqrt{2})^4}{4} - 0 + 0\right] d\theta$$
$$= \int_0^{2\pi} \left[6 - \frac{12}{4}\right] d\theta = \int_0^{2\pi} 3\, d\theta = 3\theta\Big|_0^{2\pi} = 3(2\pi) - 0 = \boxed{6\pi}$$

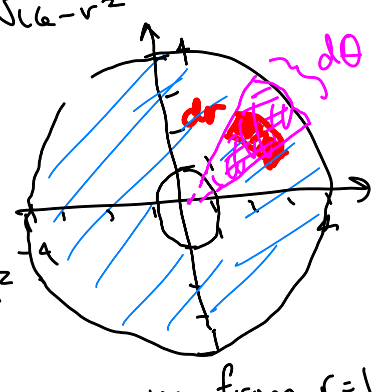
Example 9: Find the volume of the solid inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = 1$.

radius 1

$$z^2 = 16 - x^2 - y^2 \longrightarrow x^2 + y^2 + z^2 = 16$$
radius 4
$$z^2 = 16 - r^2$$
$$z = \sqrt{16 - r^2}$$

$$\text{Volume} = \int_0^{2\pi}\int_1^{4} \sqrt{16 - r^2}\; r\, dr\, d\theta$$

$u = 16 - r^2$
$du = -2r\, dr$
$-\frac{1}{2} du = r\, dr$
$r = 1 \Rightarrow u = 16 - 1^2 = 15$
$r = 4 \Rightarrow u = 16 - 4^2 = 0$

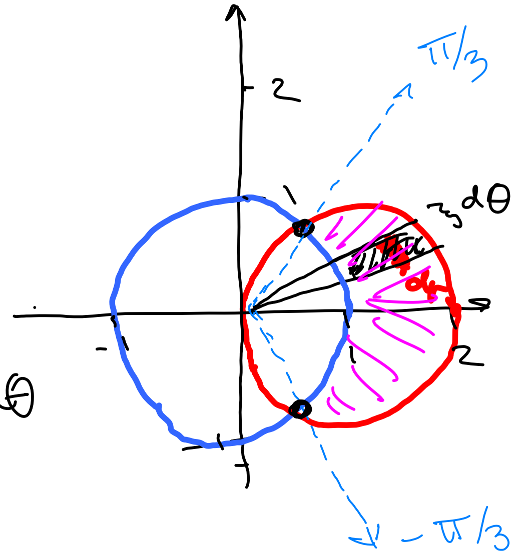


$$= -\frac{1}{2}\int_0^{2\pi}\int_{15}^{0} u^{1/2}\, du\, d\theta$$
$$= -\frac{1}{2}\int_0^{2\pi} \frac{u^{3/2}}{3/2}\Big|_{15}^{0} d\theta$$
$$= -\frac{1}{2}\cdot\frac{2}{3}\int_0^{2\pi} \left(0^{3/2} - 15^{3/2}\right) d\theta$$

dr moves from r=1 to r=4
dθ moves from 0 to 2π

$$= \frac{1}{3}\int_0^{2\pi} 15^{3/2}\, d\theta = \frac{1}{3}\cdot 15^{3/2}\,\theta\Big|_0^{2\pi}$$
$$= \frac{1}{3}\cdot 15^{3/2}(2\pi - 0) = \frac{2\pi(15)^{3/2}}{3} = \frac{2\pi\sqrt{15}\sqrt{15}\sqrt{15}}{3}$$
$$= \frac{2\pi(15)\sqrt{15}}{3} = \boxed{10\pi\sqrt{15}}$$

Example 10: Find the area of the region inside the graph of $r = 2\cos\theta$ and outside the graph of $r = 1$.



dr runs from r=1 to r= 2cosθ

Where do they intersect?

set r's equal: $1 = 2\cos\theta$

$$\frac{1}{2} = \cos\theta$$

$$\frac{\pi}{3} = \theta$$

also, $-\frac{\pi}{3}, \frac{5\pi}{3}$

$$\text{Area} \int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} r\,dr\,d\theta = 2\int_{0}^{\pi/3}\int_{1}^{2\cos\theta} r\,dr\,d\theta$$

$$= 2\int_{0}^{\pi/3} \frac{r^2}{2}\Big|_{1}^{2\cos\theta} d\theta$$

$$= \int_{0}^{\pi/3} r^2\Big|_{1}^{2\cos\theta} d\theta = \int_{0}^{\pi/3} (4\cos^2\theta - 1^2)\,d\theta$$

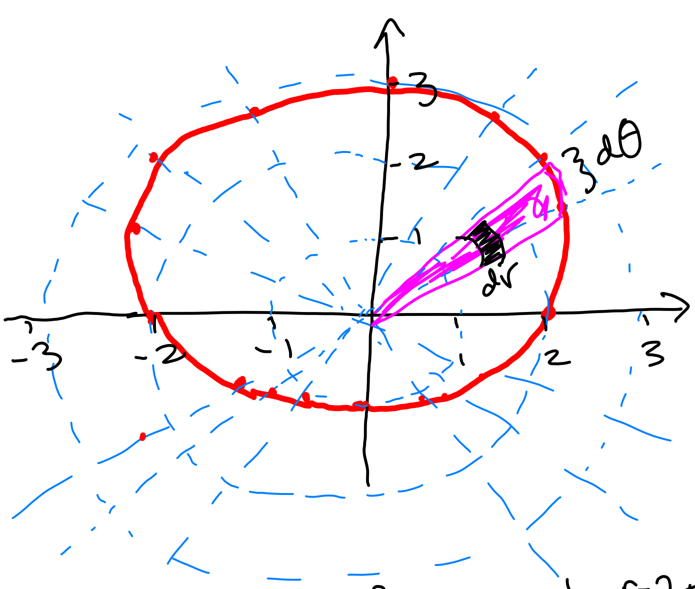
$$= 4\int_{0}^{\pi/3} \cos^2\theta\,d\theta - \int_{0}^{\pi/3} 1\,d\theta = 4\int_{0}^{\pi/3} \frac{1}{2}[1+\cos 2\theta]\,d\theta - \theta\Big|_{0}^{\pi/3}$$

$$= 2\int_{0}^{\pi/3} [1+\cos 2\theta]\,d\theta - \frac{\pi}{3} + 0 = \left[2\theta + 2\left(\frac{1}{2}\right)\sin 2\theta\right]\Big|_{0}^{\pi/3} - \frac{\pi}{3}$$

$$= \frac{2\pi}{3} + \sin\left(\frac{2\pi}{3}\right) - \frac{\pi}{3}$$

$$= \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

Example 11: Find the area of the region enclosed by the graph of $r = 2 + \sin\theta$.



θ	$r = 2 + \sin\theta$
0	$2 + \sin 0 = 2 + 0 = 2$
$\frac{\pi}{6}$	$2 + \sin\frac{\pi}{6} = 2 + \frac{1}{2} = 2.5$
$\frac{\pi}{4}$	$2 + \sin\frac{\pi}{4} = 2 + \frac{\sqrt{2}}{2} \approx 2.7$
$\frac{\pi}{3}$	$2 + \sin\frac{\pi}{3} = 2 + \frac{\sqrt{3}}{2} \approx 2.87$
$\frac{\pi}{2}$	$2 + \sin\frac{\pi}{2} = 2 + 1 = 3$

Quadrant II - similar by symmetry

π	$2 + \sin\pi = 2 + 0 = 2$
$\frac{7\pi}{6}$	$2 + \sin\frac{7\pi}{6} = 2 - \frac{1}{2} = 1.5$
$\frac{5\pi}{4}$	$2 + \sin\frac{5\pi}{4} = 2 - \frac{\sqrt{2}}{2} \approx\ \approx 1.29$
$\frac{4\pi}{3}$	$2 + \sin\frac{4\pi}{3} = 2 - \frac{\sqrt{3}}{2} \approx 1.13$
$3\pi/2$	$2 + \sin\frac{3\pi}{2} = 2 - 1 = 1$

dr runs from $r=0$ to $r = 2+\sin\theta$

dθ runs from $\theta = 0$ to $\theta = 2\pi$

integral on next page

Ex 11 cont'd

$$\text{Area} = \int_0^{2\pi}\int_0^{2+\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2}\Big|_0^{2+\sin\theta} d\theta = \frac{1}{2}\int_0^{2\pi}\left[(2+\sin\theta)^2 - 0^2\right] d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(4 + 4\sin\theta + \sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\left[4\theta - 4\cos\theta\right]\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= \frac{1}{2}\left[4(2\pi) - 4\cos 2\pi - 0 + 4\cos 0\right] + \frac{1}{2}\int_0^{2\pi} \frac{1}{2}\left[1 - \cos 2\theta\right] d\theta$$

$$= \frac{1}{2}\left[8\pi \cancel{- 4(1)} \cancel{+ 4(1)}\right] + \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]\Big|_0^{2\pi}$$

$$= 4\pi + \frac{1}{4}\left[2\pi - \frac{1}{2}\sin 4\pi - 0 + \frac{1}{2}\sin 0\right]$$

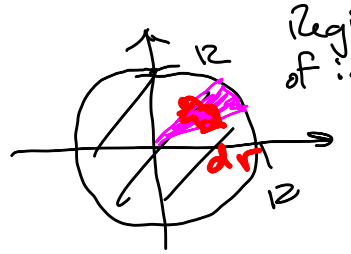
$$= 4\pi + \frac{\pi}{2} - 0 = \frac{8\pi}{2} + \frac{\pi}{2} = \boxed{\frac{9\pi}{2}}$$

Example 12: Find the volume of the sphere of radius *R*.

Trace in xy plane:

$z = 0$

$x^2 + y^2 = R^2$



Region of integration Ω

$$x^2 + y^2 + z^2 = R^2$$

$$z = \pm\sqrt{R^2 - x^2 - y^2}$$

Top half: $z = \sqrt{R^2 - x^2 - y^2}$

Volume of top half:

$$V = \iint_\Omega \sqrt{R^2 - x^2 - y^2}\, dA = \iint_\Omega \sqrt{R^2 - \underbrace{(x^2+y^2)}_{r^2}}\, dA$$

$$= \int_0^{2\pi}\int_0^R \sqrt{R^2 - r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^R (R^2 - r^2)^{1/2}\, r\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi}\int_{R^2}^0 u^{1/2}\,du\,d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} \frac{u^{3/2}}{3/2}\Bigg|_{R^2}^0 d\theta$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\int_0^{2\pi} u^{3/2}\Big|_{R^2}^0 d\theta = -\frac{1}{3}\int_0^{2\pi}\left[0^{3/2} - (R^2)^{3/2}\right] d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}(-R^3)\,d\theta = \frac{1}{3}\int_0^{2\pi} R^3\,d\theta$$

$$= \frac{1}{3}R^3\theta\Big|_0^{2\pi} = \frac{1}{3}R^3(2\pi - 0) = \frac{2\pi}{3}R^3$$ = Volume of top half

$u = R^2 - r^2$

$\frac{du}{dr} = -2r$

$du = -2r\,dr$

$-\frac{1}{2}du = r\,dr$

$r = 0 \Rightarrow u = R^2 - 0^2 = R^2$

$r = R \Rightarrow u = R^2 - R^2 = 0$

Volume of whole sphere $= 2\left(\frac{2\pi}{3}R^3\right) = \frac{4\pi}{3}R^3 = \boxed{\frac{4}{3}\pi R^3}$