14.4: Center of Mass and Moments of Inertia

Density and mass:

Lamina: a thin planar region (a thin plate—sufficiently thin that the thickness can be neglected.)

For solids, density is expressed as mass per unit volume.
For lamina, density is express as mass per unit surface area.

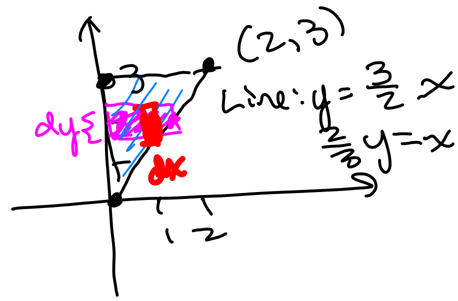
Density is usually denoted by the Greek letter rho: ρ

If the density ρ is constant, and if A represents area, then the mass of a lamina is $m = \rho A$.

> Definition: Mass of a Planar Lamina of Variable Density
>
> If ρ is a continuous density function on the lamina corresponding to a planar region R, then the mass of the lamina is
>
> $$m = \iint_R \rho(x, y)\, dA.$$

Example 1: Find the mass of the triangular lamina with vertices $(0,0)$, $(0,3)$, and $(2,3)$, given that the density at (x, y) is $\rho(x, y) = 2x + y$.

rho



$$m = \iint_R \rho(x,y)\,dA = \iint_R (2x+y)\,dA$$

$$= \int_0^3 \int_0^{(2/3)y} (2x+y)\,dx\,dy$$

$$= \int_0^3 \left[x^2 + yx\right]_0^{\frac{2}{3}y} dy$$

$$= \int_0^3 \left[\left(\tfrac{2}{3}y\right)^2 + y\left(\tfrac{2}{3}y\right) - 0\right] dy$$

$$= \int_0^3 \left[\tfrac{4}{9}y^2 + \tfrac{6}{9}y^2\right] dy = \int_0^3 \tfrac{10}{9}y^2\,dy = \tfrac{10}{9}\cdot\tfrac{y^3}{3}\Big|_0^3$$

$$= \tfrac{10}{27}\left[3^3 - 0^3\right] = \tfrac{10}{27}\cdot 27 = \boxed{10}$$

dx runs from $x = 0$ to $x = \frac{2}{3}y$

dy runs from $y=0$ to $y=3$

Moments and centers of mass:

Moment is a force's tendency to produce rotation around a particular point. To calculate moment, we multiply the force by its perpendicular distance to the point.

For a point mass on the *xy*-plane, its *moment of mass with respect to the x-axis* is the product of the mass and its perpendicular distance to the *x*-axis. Its *moment of mass with respect to the y-axis* is the product of the mass and its perpendicular distance to the *y*-axis.

For lamina, we can think of multiplying each incremental bit of mass Δm with that point's distance *d* from the axis. To get the incremental bit of mass Δm, we multiply the density times the incremental area: $\Delta m = \rho(x, y)\, dA$.

The center of mass is defined to be the point $(\bar{x}, \bar{y})$ such that $m\bar{x} = M_y$ and $m\bar{y} = M_x$.

<u>Moments of Mass of a Planar Lamina of Variable Density</u>

Suppose ρ is a continuous density function on the planar lamina *R*.

The lamina's moment of mass with respect to the *x*-axis is

$$M_x = \iint_R y\rho(x, y)\, dA.$$

The lamina's moment of mass with respect to the *y*-axis is

$$M_y = \iint_R x\rho(x, y)\, dA.$$

The <u>center of mass</u> of the lamina is $(\bar{x}, \bar{y}) = \left(\frac{M_x}{m}, \frac{M_y}{m}\right)$.

<u>Note</u>: If *R* is a plane region, we can treat it as a lamina with density $\rho = 1$. In that case, the center of mass is called the *centroid* of the region.

$$M_x = \iint_R y\,\rho(x,y)\,dA$$

Example 2: Find the center of mass for the triangular lamina in Example 1.

From example 1, $m = 10$. $\rho(x,y) = 2x+y$

$$M_x = \int_0^3\int_0^{2y/3} y\,(2x+y)\,dx\,dy = \int_0^3\int_0^{2y/3} 2xy + y^2\,dx\,dy$$

$$= \int_0^3 \left[x^2y + y^2x\right]_0^{2y/3} dy = \int_0^3 \left[\left(\tfrac{2y}{3}\right)^2 y + y^2\left(\tfrac{2y}{3}\right) - 0\right] dy$$

$$= \int_0^3 \left[\tfrac{4}{9}y^3 + \tfrac{6}{9}y^3\right] dy = \frac{10}{9}\int_0^3 y^3\,dy = \frac{10}{9}\cdot\frac{y^4}{4}\Big|_0^3$$

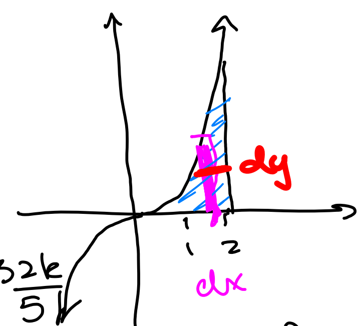
$$= \frac{10}{36}\left[3^4 - 0^4\right] = \frac{5}{18}\cdot 81 = \frac{405}{18} = \frac{45}{2}$$

$$m\bar{y} = M_x \Rightarrow \bar{y} = \frac{M_x}{m} = \frac{405/18}{10} = \frac{405}{18}\cdot\frac{1}{10} = \frac{45}{2(10)} = \frac{45}{20} = \frac{9}{4}$$

$$M_y = \int_0^3\int_0^{2y/3} x\,(2x+y)\,dx\,dy \Rightarrow \bar{x} = \frac{17}{20}$$

See next page for the step-by-step

Example 3: Find the center of mass for the region bounded by the graphs of $y = x^3$, $y = 0$, and $x = 2$, with density $\rho = kx$



$$m = \iint_A \rho(x,y)\,dA = \int_0^2\int_0^{x^3} kx\,dy\,dx$$

$$= \int_0^2 kxy\Big|_0^{x^3} dx = k\int_0^2\left[x(x^3) - x(0)\right]dx$$

$$= k\int_0^2 x^4\,dx = k\frac{x^5}{5}\Big|_0^2 = k\left(\frac{2^5}{5}\right) - 0 = \frac{32k}{5}$$

$$M_x = \int_0^2\int_0^{x^3} y\,\rho(x,y)\,dy\,dx = \int_0^2\int_0^{x^3} y\,(kx)\,dy\,dx = \int_0^2 kx\frac{y^2}{2}\Big|_0^{x^3}$$

$$= \int_0^2 kx\left(\frac{(x^3)^2}{2} - \frac{0^2}{2}\right)dx = \frac{k}{2}\int_0^2 x\cdot x^6\,dx = \frac{k}{2}\int_0^2 x^7\,dx = \frac{k}{2}\cdot\frac{x^8}{8}\Big|_0^2$$

$$= \frac{k}{16}\cdot\left(2^8 - 0^8\right) = \frac{k}{2^4}\cdot 2^8 = 2^4k = 16k$$

$$M_y = \int_0^2\int_0^{x^3} x\,\rho(x,y)\,dy\,dx = \int_0^2\int_0^{x^3} x\,(kx)\,dy\,dx = k\int_0^2\int_0^{x^3} x^2\,dy\,dx$$

$$= k\int_0^2 x^2y\Big|_0^{x^3} dx = k\int_0^2 x^2(x^3 - 0^3)\,dx = k\int_0^2 x^5\,dx$$

$$= k\frac{x^6}{6}\Big|_0^2 = k\left(\frac{2^6 - 0^6}{6}\right) = \frac{64k}{6} = \frac{32k}{3}$$

$$m\bar{y} = M_x \Rightarrow \bar{y} = \frac{M_x}{m} = \frac{16k}{\left(\frac{32k}{5}\right)} = 16k\left(\frac{5}{32k}\right) = \frac{5}{2}$$

Cont'd next Page

Example 3 cont'd:

$$m\bar{x} = M_y \Rightarrow \bar{x} = \frac{M_y}{m} = \frac{32k/3}{32k/5} = \frac{32k}{3} \cdot \frac{5}{32k} = \frac{5}{3}$$

$$\boxed{\text{Center of mass: } (\bar{x},\bar{y}) = \left(\frac{5}{3}, \frac{5}{2}\right)}$$

Example 2 continued:

$$M_y = \int_0^3 \int_0^{2y/3} x(2x+y)\,dx\,dy = \int_0^3 \int_0^{\frac{2y}{3}} (2x^2 + xy)\,dx\,dy$$

$$= \int_0^3 \left[\frac{2x^3}{3} + \frac{x^2y}{2}\right]\Bigg|_0^{2y/3} dy = \int_0^3 \left[\frac{2}{3}\cdot x^3 + \frac{y}{2}\cdot x^2\right]\Bigg|_0^{\frac{2y}{3}} dy$$

$$= \int_0^3 \left[\frac{2}{3}\left(\frac{2y}{3}\right)^3 + \frac{y}{2}\cdot\left(\frac{2y}{3}\right)^2 - 0\right] dy = \int_0^3 \left[\frac{2}{3}\cdot\frac{8y^3}{27} + \frac{y}{2}\cdot\frac{4y^2}{9}\right] dy$$

$$= \int_0^3 \left[\frac{16}{81}y^3 + \frac{2}{9}y^3\right] dy = \int_0^3 \left[\frac{16}{81}y^3 + \frac{18}{81}y^3\right] dy = \frac{34}{81}\int_0^3 y^3\,dy$$

$$= \frac{34}{81}\cdot\frac{y^4}{4}\Bigg|_0^3 = \frac{34}{81}\left[\frac{3^4}{4} - \frac{0^4}{4}\right] = \frac{34}{81}\left(\frac{81}{4}\right) = \frac{34}{4} = \frac{17}{2}$$

$$\bar{x}\,m = M_y \Rightarrow \bar{x} = \frac{M_y}{m} = \frac{17/2}{10} = \frac{17}{2}\cdot\frac{1}{10} = \frac{17}{20}$$

$$\boxed{\text{The center of mass is } (\bar{x},\bar{y}) = \left(\frac{17}{20}, \frac{9}{4}\right)}$$

Moments of inertia:

We're now familiar with the concept of moment:

<u>Moment of mass</u> (sometimes called the <u>first moment</u>) of an object about a point: measures the object's tendency to rotate about that point.

<u>Moment of inertia</u> (sometimes called the <u>second moment</u>) of an object about a point: measures the object's tendency to resist a change in rotational motion.

<u>Moments of Inertia of a Planar Lamina of Variable Density</u>

Suppose ρ is a continuous density function on the planar lamina R.

The lamina's moment of inertia with respect to the x-axis is

$$I_x = \iint_R y^2 \rho(x,y)\,dA.$$

Radius of Gyration with respect to x-axis:

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}}$$

The lamina's moment of inertia with respect to the y-axis is

$$I_y = \iint_R x^2 \rho(x,y)\,dA.$$

Radius of gyration w.r.t. y-axis: $\bar{\bar{x}} = \sqrt{\frac{I_y}{m}}$

The <u>polar moment of inertia</u> I_0 is the sum of the moments of inertia I_x and I_y. It measures the object's ~~tendency to rotate~~ resistance to change in rotation around the z-axis.

$$I_0 = \iint_R (x^2+y^2)\rho(x,y)\,dA = \iint_R r^2 \rho(x,y)\,dA.$$

$$I_0 = I_x + I_y$$

<u>Example 4:</u> Find the moments of inertia I_x, I_y, and I_0 for the triangular lamina in Example 1 and Example 2.

$$I_x = \iint_R y^2 \rho(x,y)\,dA = \int_0^3 \int_0^{2y/3} y^2(2x+y)\,dx\,dy$$

$$= \int_0^3 \int_0^{2y/3} (2xy^2 + y^3)\,dx\,dy = \int_0^3 \left(2y^2 \cdot \frac{x^2}{2} + y^3 x\right)\Big|_0^{\frac{2y}{3}} dy$$

$$= \int_0^3 y^2\left(\frac{2y}{3}\right)^2 + y^3\left(\frac{2y}{3}\right) dy = \int_0^3 \left[y^2\left(\frac{4y^2}{9}\right) + \frac{2y^4}{3}\right] dy$$

$$= \int_0^3 \left[\frac{4}{9}y^4 + \frac{6}{9}y^4\right] dy = \frac{10}{9}\int_0^3 y^4\,dy = \frac{10}{9}\cdot\frac{y^5}{5}\Big|_0^3 = \frac{2}{9}(3^5 - 0^5)$$

$$= \frac{2}{3^2}\cdot 3^5 = 2(3^3) = 2(27) = 54$$

see next page

$$I_y = \iint_R x^2 \rho(x,y)\,dA = \int_0^3 \int_0^{\frac{2y}{3}} x^2(2x+y)\,dx\,dy$$

$$= \int_0^3\int_0^{\frac{2y}{3}} (2x^3 + x^2 y)\,dx\,dy = \int_0^3 \left[\frac{\cancel{2}^{1} x^4}{\cancel{4}_{2}} + \frac{x^3 y}{3}\right]\Bigg|_0^{2y/3} dy$$

$$= \int_0^3 \left[\frac{1}{2}\left(\frac{2y}{3}\right)^4 + \frac{y}{3}\left(\frac{2y}{3}\right)^3\right]dy = \int_0^3\left[\frac{16y^4}{2\cdot 81} + \frac{y}{3}\cdot\frac{8y^3}{27}\right]dy$$

$$= \int_0^3\left[\frac{8}{81}y^4 + \frac{8}{81}y^4\right]dy = \frac{16}{81}\int_0^3 y^4\,dy = \frac{16}{81}\cdot\frac{y^5}{5}\Bigg|_0^3$$

$$= \frac{16}{405}(3^5 - 0^5) = \frac{16}{5\cdot 3^4}(3^5) = \frac{16}{5}(3) = \frac{48}{5} = 9.6$$

$$I_x = 54,\ I_y = \frac{48}{5} \Rightarrow I_0 = I_x + I_y = 54 + \frac{48}{5} = 54 + 9.6 = \boxed{63.6}$$

$$= \frac{270}{5} + \frac{48}{5} = \frac{318}{5}$$

From Example 1, $m = 10$.

So, the radius of gyration about the x-axis is

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{54}{10}} = \frac{3\sqrt{6}}{\sqrt{10}} = \frac{3\sqrt{6}\sqrt{10}}{10} = \frac{3\sqrt{60}}{10} = \frac{3\cdot 2\sqrt{15}}{10}$$

$$= \boxed{\frac{3\sqrt{15}}{5}}$$

The radius of gyration about the y-axis is

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{318/5}{10}} = \sqrt{\frac{318}{5}}\left(\frac{1}{\sqrt{10}}\right) = \frac{\sqrt{318}}{\sqrt{50}} = \frac{\sqrt{318}}{5\sqrt{2}}$$

$$= \frac{\sqrt{318}}{5\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{636}}{5\cdot 2} = \frac{\sqrt{4(159)}}{10} = \frac{2\sqrt{159}}{10} = \boxed{\frac{\sqrt{159}}{5}}$$