14.5: Surface Area

We can use a double integral to find the upper surface area of a solid defined by $z = f(x, y)$ over a region *R*.

> Definition: If *f* and its first partial derivatives are continuous on the closed region *R* in the *xy*-plane, then the area of the surface S given by $z = f(x, y)$ over R is defined as
>
> $$\text{Surface Area} = \iint_R dS$$
> $$= \iint_R \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\, dA.$$

Note:

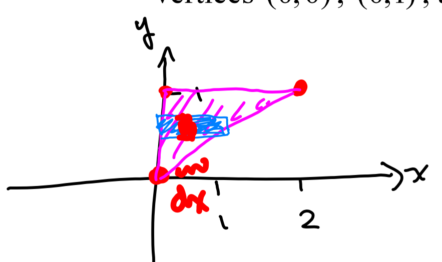
Length on *x*-axis: $\int_a^b dx$ $\qquad \int_a^b dx = \int_a^b 1\,dx = x\Big|_a^b = b-a$

Arc length in *xy*-plane: $\int_a^b \sqrt{1+[f'(x)]^2}\, dx$

Area in *xy*-plane: $\iint_R dA$

Surface area in $\mathbb{R}^3$: $\iint_R \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\, dA$

Example 1: Find the area of the surface $z = 1+3x+2y^2$ that lies above the triangle with vertices $(0,0)$, $(0,1)$, and $(2,1)$.



Line has slope $\frac{1}{2}$ and y-intercept 0.

Eqn of line: $y = \frac{1}{2}x$

$\Downarrow$

$2y = x$

$$\text{Surface Area} = \iint \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\, dA$$

$$= \int_0^1 \int_0^{2y} \sqrt{1+(3)^2+(4y)^2}\, dx\, dy$$

$$= \int_0^1 \int_0^{2y} \sqrt{10+16y^2}\, dx\, dy$$

$$= \int_0^1 x\sqrt{10+16y^2}\Big|_{x=0}^{x=2y} dy$$

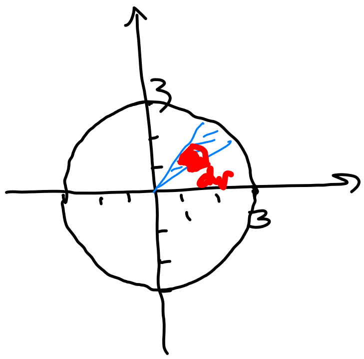
$$= \int_0^1 \left[2y\sqrt{10+16y^2} - 0\right] dy = 2\int_0^1 y(10+16y^2)^{1/2} dy = 2\left(\frac{1}{32}\right)\cdot \frac{(10+16y^2)^{3/2}}{3/2}\Big|_0^1$$

$$= \frac{1}{24}\left[(10+16(1)^2)^{3/2} - (10+0)^{3/2}\right]$$

$$\boxed{= \frac{1}{24}\left[26^{3/2} - 10^{3/2}\right]}$$

≈ 4.2063

Example 2: Find the area of the surface $f(x,y)=12+2x-3y$ that lies above the region R bounded by the graph of $x^2+y^2\le 9$.



$$S=\iint_R \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\,dA$$

$$=\int_0^{2\pi}\int_0^3 \sqrt{1+(2)^2+(-3)^2}\;r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^3 \sqrt{14}\;r\,dr\,d\theta=\int_0^{2\pi}\sqrt{14}\frac{r^2}{2}\Big|_0^3\,d\theta=\sqrt{14}\int_0^{2\pi}\left(\frac{9}{2}-0\right)d\theta=\sqrt{14}\left(\frac{9}{2}\theta\right)\Big|_0^{2\pi}$$

$$=\sqrt{14}\left(\frac{9}{2}\right)(2\pi-0)=\boxed{9\pi\sqrt{14}}$$

Example 3: Find the surface area of the part of the sphere $x^2+y^2+z^2=4$ that lies above the plane $z=1$.

$$z^2=4-x^2-y^2$$

$z=\pm\sqrt{4-x^2-y^2}$. Above plane $z=1$, we have $z=\sqrt{4-x^2-y^2}$
$=(4-x^2-y^2)^{1/2}$

$$\frac{\partial z}{\partial x}=\frac{1}{2}(4-x^2-y^2)^{-1/2}(-2x)=\frac{-x}{\sqrt{4-x^2-y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{1}{2}(4-x^2-y^2)(-2y)=\frac{-y}{\sqrt{4-x^2-y^2}}$$

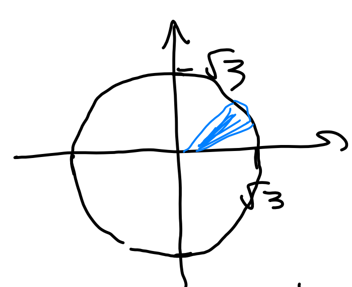
So integrand $=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}}=\sqrt{\frac{4-x^2-y^2+x^2+y^2}{4-x^2-y^2}}$

$$=\sqrt{\frac{4}{4-x^2-y^2}}=\frac{\sqrt{4}}{\sqrt{4-x^2-y^2}}=\frac{2}{\sqrt{4-x^2-y^2}}=2(4-x^2-y^2)^{-1/2}$$

Put $z=1$ into $x^2+y^2+z^2=4$:

$$x^2+y^2+1^2=4$$

$x^2+y^2=3 \Rightarrow$ Region is a circle of radius $\sqrt{3}$.



$$S=\iint_R 2(4-x^2-y^2)^{-1/2}\,dA$$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}} 2(4-x^2-y^2)^{-1/2}\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}} 2(4-\underbrace{(x^2+y^2)}_{r^2})^{-1/2}\,r\,dr\,d\theta=2\int_0^{2\pi}\int_0^{\sqrt{3}}(4-r^2)^{-1/2}\,r\,dr\,d\theta$$

(next page)

$$= 2\left(\frac{1}{-2}\right)\int_0^{2\pi} \frac{(4-r^2)^{1/2}}{1/2}\Bigg|_0^{\sqrt{3}} d\theta = -2\int_0^{2\pi}\left[(4-(\sqrt{3})^2)^{1/2} - (4-0^2)^{1/2}\right] d\theta$$

$$= -2\int_0^{2\pi}\left[1^{1/2} - 4^{1/2}\right] d\theta = -2\int_0^{2\pi}(1-2)\,d\theta = -2\int_0^{2\pi}(-1)\,d\theta = 2\,\theta\Big|_0^{2\pi}$$

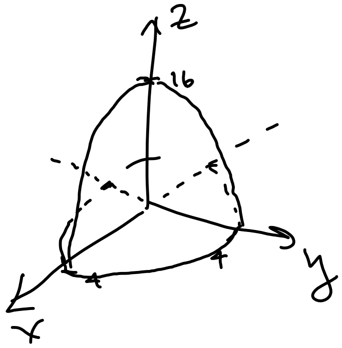
$$2(2\pi - 0) = \boxed{4\pi}$$

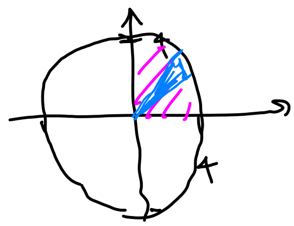
Example 4: Find the surface area of the portion of the paraboloid $z = 16 - x^2 - y^2$ that lies in the first octant.

Trace in xy-plane:

$z = 0 \Rightarrow 0 = 16 - x^2 - y^2$

$x^2 + y^2 = 16$ (circle of radius 4)





$$S = \int_0^{\pi/2}\int_0^4 \sqrt{1+(-2x)^2+(-2y)^2}\; r\,dr\,d\theta = \int_0^{\pi/2}\int_0^4 r\sqrt{1+4\underbrace{(x^2+y^2)}_{r^2}}\;dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^4 r(1+4r^2)^{1/2}\,dr\,d\theta \qquad = \frac{1}{8}\int_0^{\pi/2} \frac{(1+4r^2)^{3/2}}{3/2}\Bigg|_0^4 d\theta$$

$$= \frac{1}{8}\cdot\frac{2}{3}\int_0^{\pi/2}\left[(1+4(4)^2)^{3/2} - (1+4(0)^2)^{3/2}\right] d\theta$$

$$= \frac{2}{24}\int_0^{\pi/2}\left[65^{3/2} - 1^{3/2}\right] d\theta = \frac{1}{12}(65^{3/2}-1)\int_0^{\pi/2} d\theta$$

$$= \frac{1}{12}(65^{3/2}-1)\,\theta\Big|_0^{\pi/2} = \frac{1}{12}(65^{3/2}-1)\left(\frac{\pi}{2}-0\right) = \boxed{\frac{\pi}{24}(65^{3/2}-1)}$$

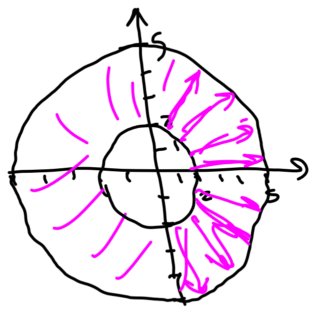
$z = (25 - x^2 - y^2)^{1/2}$

Example 5: Find the surface area of the portion of the hemisphere $z = \sqrt{25 - x^2 - y^2}$ that lies outside the cylinder $x^2 + y^2 = 4$.

Sphere: $z^2 + x^2 + y^2 = 25$ (radius 5)

cylinder has radius 2.

Projects to this Area:



dr runs from $r = 2$ to $r = 5$

$d\theta$ runs from 0 to 2π

$$z_x = \frac{1}{2}(25 - x^2 - y^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{25 - x^2 - y^2}}$$

$$z_y = \frac{1}{2}(25 - x - y^2)^{-1/2}(-2y) = \frac{-y}{\sqrt{25 - x^2 - y^2}}$$

$$S = \int_0^{2\pi}\int_2^5 \sqrt{1 + \left(\frac{-x}{\sqrt{25 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{25 - x^2 - y^2}}\right)^2}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^5 \sqrt{\frac{25 - x^2 - y^2 + x^2 + y^2}{25 - x^2 - y^2}}\; r\,dr\,d\theta = \int_0^{2\pi}\int_2^5 \sqrt{\frac{25}{25 - \underbrace{(x^2 + y^2)}_{r^2}}}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^5 \frac{5}{\sqrt{25 - r^2}}\, r\,dr\,d\theta = 5\int_0^{2\pi}\int_2^5 r(25 - r^2)^{-1/2}\,dr\,d\theta$$

⋮

$$= \boxed{10\pi\sqrt{21}}$$

Homework Qs

14.3 #23

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} xy \, dy \, dx$$

$y = \sqrt{2x - x^2}$

$y^2 = 2x - x^2$

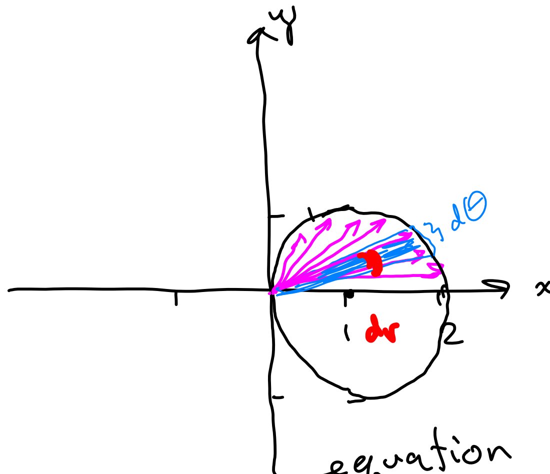
$x^2 - 2x + y^2 = 0$

$(x^2 - 2x + \underline{1}) + y^2 = 0 + \underline{1}$

$\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$

$(x-1)^2 + y^2 = 1$

Circle of radius 1, center $(1,0)$



equation of this circle in polar is $r = 2\cos\theta$

$$\int_0^{\pi/2} \int_0^{2\cos\theta} xy \underbrace{r \, dr \, d\theta}_{dA}$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} (r\cos\theta)(r\sin\theta) r \, dr \, d\theta$$

$\left.\begin{aligned} r &= a\cos n\theta \\ r &= a\sin n\theta \end{aligned}\right\}$ "flowers/roses"

n odd $\Rightarrow$ n petals

n even $\Rightarrow$ $2n$ petals