14.6: Triple Integrals and Applications

Definition: (Triple Integral)

Suppose $f(x, y, z)$ is continuous over a bounded solid region Q in $\mathbb{R}^3$. Also suppose that Q is partitioned into n three-dimensional boxes, in such a way that the norm of the partition (diagonal of the largest box, denoted $\|\Delta\|$) approaches 0 as the number of boxes approaches infinity. (In other words, $\|\Delta\| \to 0$ as $n \to \infty$). Then the triple integral of f over Q is

$$\iiint_Q f(x, y, z)\, dV \lim_{\|\Delta\| \to 0, n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta V_i,$$

where ΔV_i is the volume of the ith box, and (x_i, y_i, z_i) is any point in the ith box (provided this limit exists).

The volume of the solid region Q is

$$\text{Volume} = \iiint_Q dV.$$

The properties of single and double integrals generally carry over to triple integrals:

$$\iiint_Q cf(x, y, z)\, dV = c \iiint_Q f(x, y, z)\, dV$$

$$\iiint_Q [f(x, y, z) + g(x, y, z)]\, dV = \iiint_Q f(x, y, z)\, dV + \iiint_Q g(x, y, z)\, dV$$

$$\iiint_Q f(x, y, z)\, dV = \iiint_{Q_1} f(x, y, z)\, dV + \iiint_{Q_2} f(x, y, z)\, dV,$$

where Q is the union of two nonoverlapping solid subregions Q_1 and Q_2.

We evaluate triple integrals by converting them into iterated integrals. When we rewrite dV in terms of x, y, and z, there are six possible orders of integration:

$$\begin{array}{lll} dxdydz & dydxdz & dzdxdy \\ dxdzdy & dydzdx & dzdydx \end{array}$$

Fubini's theorem can be extended to show that all these orders of integration will give the same result, provided the boundary functions are continuous.

<u>Fubini's Theorem</u>:

Suppose $f(x,y,z)$ is continuous on the solid region Q. Suppose also that Q is defined by $a \le x \le b$, $h_1(x) \le y \le h_2(x)$, and $g_1(x,y) \le z \le g_2(x,y)$, where g_1, g_2, h_1, and h_2 are continuous functions. Then

$$\iiint_Q f(x,y,z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

If we swap around the roles and boundaries for the variables, we can write an equivalent version of Fubini's theorem for all six possible orders of integration.

When we evaluate the innermost integral, we hold two of the variables constant. In the second integration, one of the remaining variables will be held constant, just as we did when evaluating double integrals.

Example 1: Evaluate the iterated integral $\int_1^4 \int_1^{e^2} \int_0^{1/(xz)} \ln z\,dy\,dz\,dx$.

$$\int_1^4 \int_1^{e^2} \int_0^{\frac{1}{xz}} (\ln z)\,dy\,dz\,dx = \int_1^4 \int_1^{e^2} (\ln z)(y)\Big|_{y=0}^{y=\frac{1}{xz}} dz\,dx$$

$$= \int_1^4 \int_1^{e^2} (\ln z)\left(\frac{1}{xz} - 0\right) dz\,dx = \int_1^4 \int_1^{e^2} \frac{1}{xz} \ln z\,dz\,dx$$

$$= \int_1^4 \frac{1}{x} \int_1^{e^2} \frac{1}{z} \ln z\,dz\,dx = \int_1^4 \frac{1}{x} \int_{u=0}^{u=2} u\,du\,dx = \int_1^4 \frac{u^2}{2}\Big|_0^2 \frac{1}{x}\,dx$$

$$= \int_1^4 \left(\frac{2^2}{2} - \frac{0^2}{2}\right)\frac{1}{x}\,dx$$

$u = \ln z$

$\frac{du}{dz} = \frac{1}{z}$

$du = \frac{1}{z}\,dz$

$z = 1 \Rightarrow u = \ln 1 = 0$

$z = e^2 \Rightarrow u = \ln e^2 = 2$

$$= \int_1^4 2\left(\frac{1}{x}\right)dx$$

$$= 2\ln|x|\,\Big|_1^4$$

$$= 2\ln 4 - 2\ln 1$$

$$= 2\ln 4 - 0 = \boxed{2\ln 4}$$

Example 2: Evaluate the iterated integral $\int_0^{\pi/2} \int_0^{y/2} \int_0^{1/y} \sin y\,dz\,dx\,dy$.

$$\int_0^{\pi/2} \int_0^{y/2} \int_0^{1/y} \sin y\,dz\,dx\,dy$$

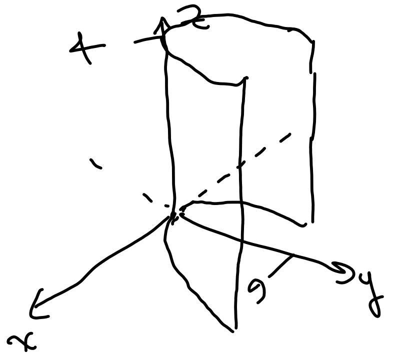
$$= \int_0^{\pi/2} \int_0^{y/2} z \sin y\Big|_{z=0}^{z=\frac{1}{y}} dx\,dy = \int_0^{\pi/2} \int_0^{y/2} \left[\frac{1}{y}\sin y - 0\sin y\right] dx\,dy$$

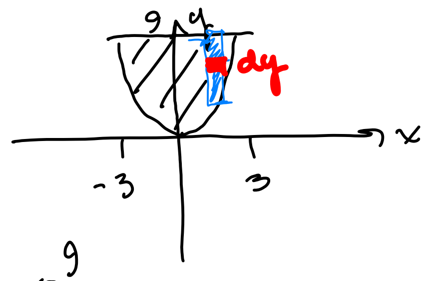
$$= \int_0^{\pi/2} \int_0^{y/2} \frac{1}{y}\sin y\,dx\,dy = \int_0^{\pi/2} x\left(\frac{1}{y}\sin y\right)\Big|_{x=0}^{x=y/2} dy = \int_0^{\pi/2} \left[\frac{y}{2}\cdot\frac{1}{y}\sin y - 0\right] dy$$

$$= \int_0^{\pi/2} \frac{1}{2}\sin y\,dy = -\frac{1}{2}\cos y\Big|_0^{\pi/2}$$

$$= -\frac{1}{2}\cos\frac{\pi}{2} + \frac{1}{2}\cos(0) = 0 + \frac{1}{2}(1) = \boxed{\frac{1}{2}}$$

Example 3: Find the volume of the solid bounded by the graphs of $y = x^2$ and the planes $y = 9$, $z = 0$, and $z = 4$.





$$V = \int_0^4 \int_{-3}^{3} \int_{x^2}^{9} dy\, dx\, dz$$

$$= \int_0^4 \int_{-3}^{3} y \Big|_{y=x^2}^{y=9} dx\, dz = \int_0^4 \int_{-3}^{3} (9 - x^2)\, dx\, dz = \int_0^4 \left(9x - \frac{x^3}{3}\right)\Big|_{-3}^{3} dz$$

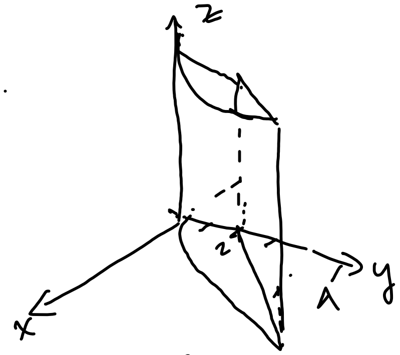
$$= \int_0^4 \left[\left(9(3) - \frac{3^3}{3}\right) - \left(9(-3) - \frac{(-3)^3}{3}\right)\right] dz = \int_0^4 [27 - 9 + 27 - 9]\, dz = \int_0^4 36\, dz$$

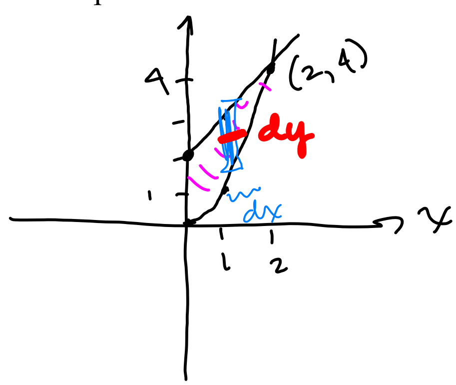
$$= 36\, z \Big|_0^4 = 36(4-0) = \boxed{144}$$

Example 4: Find the volume of the first-octant portion of the solid bounded by the graphs of $y = x^2$, $z = x$, and $y = x + 2$.

$m = 1$

y-int $= 2$





$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

$$V = \int_0^2 \int_{x^2}^{x+2} \int_0^x dz\, dy\, dx = \int_0^2 \int_{x^2}^{x+2} z \Big|_0^x dy\, dx$$

$x^2 \le y \le x + 2$

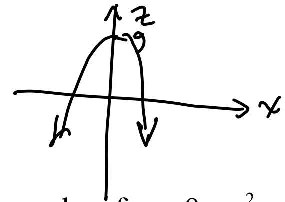
$0 \le z \le x$

$0 \le x \le 2$

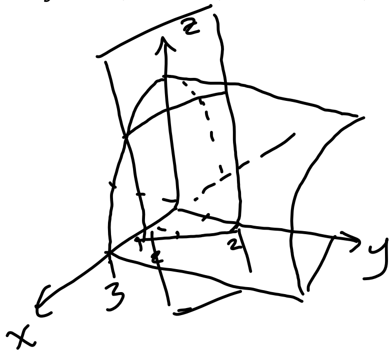
$$= \int_0^2 \int_{x^2}^{x+2} x\, dy\, dx = \int_0^2 xy \Big|_{y=x^2}^{y=x+2} dx$$

$$= \int_0^2 [x(x+2) - x(x^2)]\, dx = \int_0^2 [x^2 + 2x - x^3]\, dx = \frac{x^3}{3} + x^2 - \frac{x^4}{4} \Big|_0^2$$

$$= \frac{2^3}{3} + 2^2 - \frac{2^4}{4} = \frac{8}{3} + 4 - 4 = \boxed{\frac{8}{3}}$$



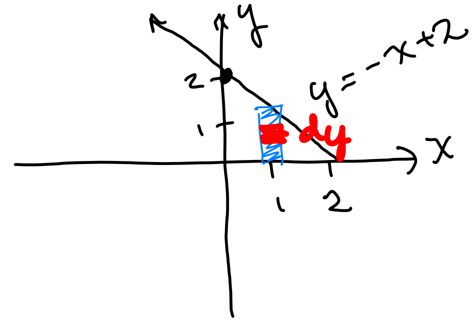
Example 5: Find the volume of the solid bounded by the graphs of $z = 9 - x^2$, $y = -x + 2$, $y = 0$, $z = 0$, and $x = 0$, with $x \geq 0$.



$z = 9 - x^2$

$z = 0 \Rightarrow 0 = 9 - x^2$

$x = \pm 3$



$$V = \int_0^2 \int_0^{-x+2} \int_0^{9-x^2} dz\, dy\, dx$$

$$= \boxed{\frac{50}{3}}$$

Example 6: Rewrite the iterated integral $\int_0^6 \int_0^{(6-x)/2} \int_0^{(6-x-2y)/3} dz\, dy\, dx$ using the order $dy\, dx\, dz$.

$$\int_0^6 \int_0^{\frac{6-x}{2}} \int_0^{\frac{6-x-2y}{3}} dz\, dy\, dx$$

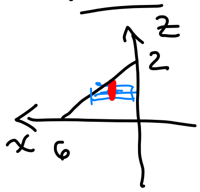
$$\boxed{\int_0^2 \int_0^{6-3z} \int_0^{\frac{1}{2}(6-x-3z)} dy\, dx\, dz}$$

$0 \leq z \leq \frac{1}{3}(6-x-2y)$

$0 \leq y \leq \frac{1}{2}(6-x)$

$0 \leq x \leq 6$

Base: dA



Look at $z = \frac{1}{3}(6-x-2y)$

mult. by 3: $3z = 6 - x - 2y$

$x + 2y + 3z = 6$

y goes from $y=0$ (xz-plane) to the surface $x+2y+3z=6$

Solve for y: $2y = 6 - x - 3z$

$y = \frac{1}{2}(6-x-3z) = \frac{6-x-3z}{2}$

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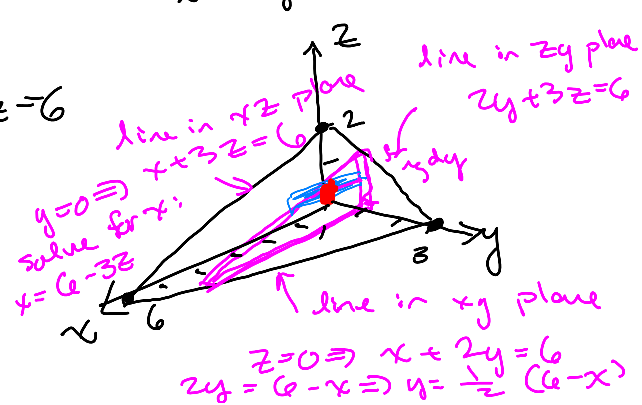


Write in order $dx\,dz\,dy$

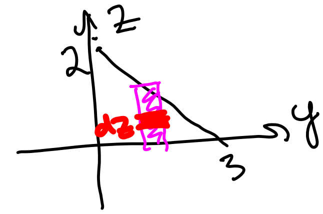
Ex 6 cont'd

$$\int_0^3 \int_0^{\frac{1}{3}(6-2y)} \int_0^{6-2y-3z} dx\,dz\,dy$$

solve plane for x: $x+2y+3z=6$

$x = 6-2y-3z$

Base: dA in yz-plane



dz runs from $z=0$ to line $2y+3z=6$

solve line for z:

$3z = 6-2y$

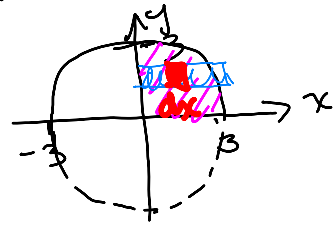
$z = \frac{1}{3}(6-2y)$

Example 7: Rewrite the iterated integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} dz\, dy\, dx$ using the order $dz\,dx\,dy$.

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} dz\, dy\, dx$$

$0 \le z \le 6-x-y$ (plane

original ~~Given~~ order has $dA = dy\,dx$



$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{6-x-y} dz\, dx\, dy$$

dx runs from $x=0$ to right half of circle

$x^2 = 9 - y^2$

$x = \pm\sqrt{9-y^2}$ right half is $x = +\sqrt{9-y^2}$

$0 \le y \le \sqrt{9-x^2}$

top half of $y^2 + x^2 = 9$

Example 8: Rewrite the iterated integral $\int_0^1 \int_0^{x^2} \int_0^{y} f(x,y,z)\, dz\, dy\, dx$ in all the other orders of integration.

$$\int_0^1 \int_0^{x^2} \int_0^{y} f(x,y,z)\, dz\, dy\, dx$$

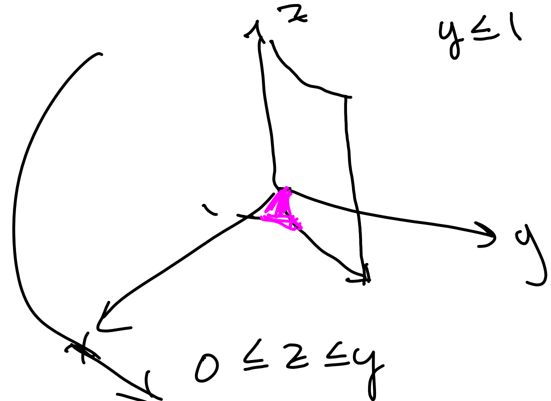
Note: $0 \le z \le y$

$0 \le y \le x^2 \Rightarrow 0 \le \sqrt{y} \le x$

$0 \le x \le 1$ $\quad \underbrace{y \le x^2,\ x \le 1}_{y \le 1}$

$$\int_0^1 \int_0^{y} \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dz\, dy$$

$$\int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\, dy\, dx\, dz$$



$0 \le z \le y$

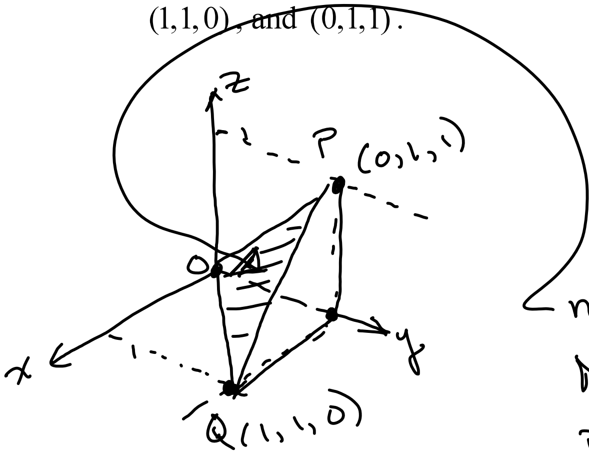
$0 \le \sqrt{y} \le x$

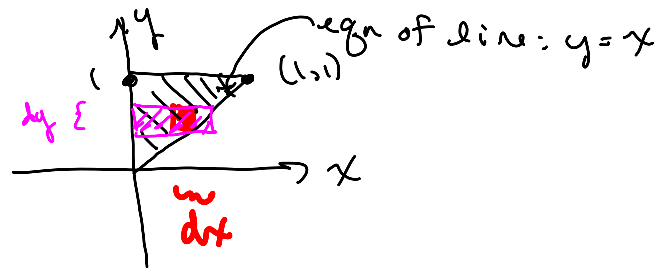
$$\int_0^1 \int_{\sqrt{y}}^1 \int_0^{y} f(x,y,z)\, dz\, dx\, dy$$

$$\int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dy\, dz$$

$$\int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z)\, dy\, dz\, dx$$

Example 9: Evaluate $\iiint_E xz\, dV$, where E is the tetrahedron with vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, and $(0,1,1)$.





need equation of this plane
Need to find 2 vectors in the plane and take their cross product.

$$\overrightarrow{PO} = \langle 0,1,1 \rangle$$
$$\overrightarrow{QO} = \langle 1,1,0 \rangle$$

$$\vec{n} = \overrightarrow{PO} \times \overrightarrow{QO} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, 1, -1 \rangle$$

Using point $O(0,0,0)$, which is on the plane:

$$-1(x-0) + 1(y-0) - 1(z-0) = 0$$
$$-x + y - z = 0 \Rightarrow z = -x + y \text{ (equation of plane)}$$

$$\iiint_E xz\, dV = \int_0^1 \int_0^y \int_0^{-x+y} xz \; dz\, dx\, dy = \int_0^1 \int_0^y x \cdot \frac{z^2}{2} \Big|_0^{y-x} dx\, dy$$

$$= \int_0^1 \int_0^y \frac{1}{2} x\left[(y-x)^2 - 0^2\right] dx\, dy = \frac{1}{2} \int_0^1 \int_0^y x(y^2 - 2xy + x^2)\, dx\, dy$$

$$= \frac{1}{2} \int_0^1 \int_0^y \left[xy^2 - 2x^2y + x^3\right] dx\, dy = \frac{1}{2} \int_0^1 \left[y^2 \cdot \frac{x^2}{2} - 2y \cdot \frac{x^3}{3} + \frac{x^4}{4}\right]\Big|_0^y dy$$

$$= \frac{1}{2} \int_0^1 \left[\frac{1}{2}y^4 - \frac{2}{3}y^4 + \frac{1}{4}y^4 - 0\right] dy = \frac{1}{2} \int_0^1 \left[\frac{6}{12}y^4 - \frac{8}{12}y^4 + \frac{3}{12}y^4\right] dy$$

$$= \frac{1}{2} \int_0^1 \frac{1}{12} y^4\, dy = \frac{1}{24} \cdot \frac{y^5}{5} \Big|_0^1 = \frac{1}{120}(1^5 - 0^5) = \boxed{\frac{1}{120}}$$

Moments and centers of mass:

If Q is a solid region with density function $\rho(x, y, z)$, then the mass of Q is

$$m = \iiint_Q \rho(x, y, z)\, dV .$$

The first moments about the *yz*-plane, the *xz*-plane, and the *xy*-plane are given by:

$$M_{yz} = \iiint_Q x\rho(x, y, z)\, dV$$
$$M_{xz} = \iiint_Q y\rho(x, y, z)\, dV$$
$$M_{xy} = \iiint_Q z\rho(x, y, z)\, dV$$

$(\bar{x}, \bar{y}, \bar{z})$ is the point at which $m\bar{x} = M_{yz}$, $m\bar{y} = M_{xz}$, $m\bar{z} = M_{xy}$

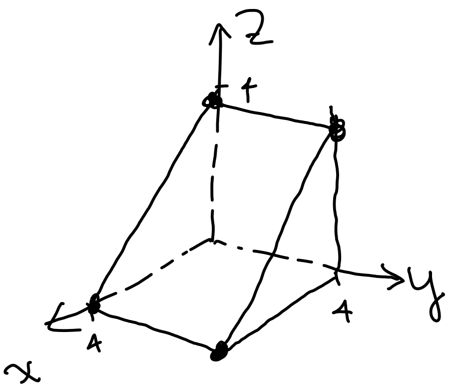
The center of mass of the solid Q is $(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x} = \frac{M_{yz}}{m}$, $\bar{y} = \frac{M_{xz}}{m}$, and $\bar{z} = \frac{M_{xy}}{m}$.

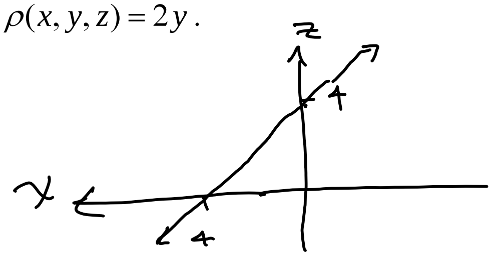
The moments of inertia (second moments) about the *x*-axis, *y*-axis, and *z*-axis are

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)\, dV$$
$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)\, dV$$
$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)\, dV$$

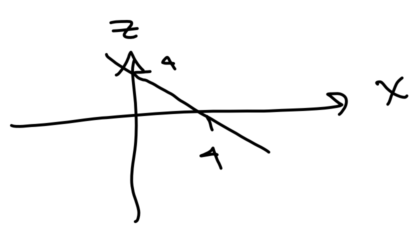
We'll skip finding the moments of inertia for solids.

Example 10: Find the mass and center of mass of the solid bounded by the graphs of $z = 4 - x$, $z = 0$, $x = 0$, $y = 0$, and $y = 4$, with density function $\rho(x, y, z) = 2y$.





$z = -x + 4$



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Ex 10 cont'd:

Find mass: $m = \iiint_E \rho(x,y,z)\,dV$

$$= \int_0^4 \int_0^4 \int_0^{4-x} 2y \, dz\, dy\, dx = \int_0^4 \int_0^4 2yz \Big|_0^{4-x} dy\, dx$$

$$= \int_0^4 \int_0^4 2y(4-x)\,dy\,dx = \int_0^4 \int_0^4 (8y - 2xy)\,dy\,dx$$

$$= \int_0^4 \left(\frac{8y^2}{2} - \frac{2xy^2}{2}\right)\Big|_0^4 dx = \int_0^4 (4y^2 - xy^2)\Big|_0^4 dx$$

$$= \int_0^4 (4(4)^2 - x(4)^2 - 0)\,dx = \int_0^4 (64 - 16x)\,dx = \left(64x - \frac{16x^2}{2}\right)\Big|_0^4$$

$$= 64(4) - 8(4)^2 - 0 = 256 - 128 = 128 \quad \text{(mass)}$$

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV = \int_0^4 \int_0^4 \int_0^{4-x} 2xy\,dz\,dy\,dx = \frac{512}{3}$$

$$M_{xz} = \int_0^4 \int_0^4 \int_0^{4-x} 2y^2\,dz\,dy\,dx = \frac{1024}{3}$$

$$M_{xy} = \int_0^4 \int_0^4 \int_0^{4-x} 2yz\,dz\,dy\,dx = \frac{512}{3}$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{512/3}{128} = \frac{4}{3}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1024/3}{128} = \frac{8}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{512/3}{128} = \frac{4}{3}$$

Center of mass:

$$\boxed{(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{4}{3}, \frac{8}{3}, \frac{4}{3}\right)}$$