14.7: Triple Integrals in Other Coordinates

Cylindrical coordinates:

If $f(x, y, z)$ is a continuous function on the solid *Q*, we can write the triple integral as

$$\iiint_Q f(x, y, z)\, dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz \right] dA,$$

where the double integral is evaluated over region *R* in the *xy*-plane. If polar coordinates are used, we write $dA = r\, dr\, d\theta$ and the triple integral is rewritten as an iterated integral in *z*, *r*, and θ:

$$\iiint_Q f(x, y, z)\, dV = \int_{\theta_1}^{\theta_1} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$

Fubini's Theorem still applies, allowing us to rearrange the order of integration.

<u>Example 1:</u> Find the volume of the solid bounded by the paraboloid $z = 36 - x^2 - y^2$ and the plane $z = 0$.

Example 2: Find the volume of the solid bounded by the paraboloid $z = 10 - 3x^2 - 3y^2$ and the plane $z = 4$. (Same problem as Example 2 in Section 14.3.)

Example 3: Evaluate the iterated integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} dz\, dy\, dx$. (Hint: Rewrite using cylindrical coordinates.)

<u>Example 4:</u> Find the volume of solid inside the graph of $x^2 + y^2 + z^2 = 16$ and outside the graph of $z = \sqrt{x^2 + y^2}$.

Example 5: Find the volume of the solid cut from the sphere $x^2 + y^2 + z^2 = 4$ by the cylinder $r = 2\sin\theta$.

Spherical coordinates:

Recall: In spherical coordinates, we write point P as (ρ, θ, φ), where ρ is the distance from the origin O to P, θ is the same angle used in cylindrical coordinates (for $r \geq 0$), and φ is the angle between $\overrightarrow{OP}$ and the positive z-axis.

Equations relating spherical coordinates to rectangular and cylindrical coordinates:

$x = \rho \sin\phi \cos\theta$ $\qquad$ $y = \rho \sin\phi \sin\theta$ $\qquad$ $z = \rho \cos\phi$

$x^2 + y^2 + z^2 = \rho^2$ $\qquad$ $\tan\theta = \dfrac{y}{x}$ $\qquad$ $\phi = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$

$r^2 = \rho^2 \sin^2\phi$ $\qquad$ $\rho = \sqrt{r^2 + z^2}$

If we start with $\iiint_Q f(x, y, z)\, dV$, the volume increment dV can be written as $\rho^2 \sin\varphi \, d\rho \; d\varphi \, d\theta$.

So, our triple integral becomes

$$\iiint_Q f(x, y, z)\, dV = \int_{\theta_1}^{\theta_1} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(x, y, z) \rho^2 \sin\varphi \, d\rho \; d\varphi \, d\theta \; .$$

Example 6: Find the volume of the sphere $x^2 + y^2 + z^2 = R^2$ using spherical coordinates.

Example 7: Rewrite the iterated integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$ in both cylindrical and spherical coordinates. Choose the easier form to integrate.

Example 8: Rewrite the iterated integral $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\; dz\, dy\, dx$ in both cylindrical and spherical coordinates.

Example 9: Find the volume of the solid bounded below by the upper half of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$.

Example 10: Set up integrals in cylindrical and spherical coordinates to evaluate $\iiint_H (x^2 + y^2)\, dV$, where H is the hemispherical region that lies above the *xy*-plane and below the sphere $x^2 + y^2 + z^2 = 1$.