14.7: Triple Integrals in Other Coordinates

Cylindrical coordinates:

If $f(x,y,z)$ is a continuous function on the solid Q, we can write the triple integral as

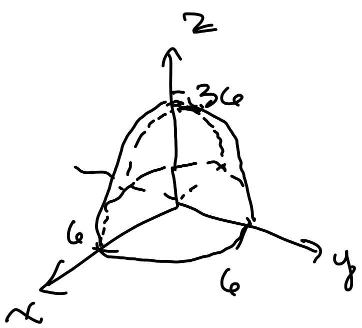
$$\iiint_Q f(x,y,z)\,dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz \right] dA,$$

where the double integral is evaluated over region R in the xy-plane. If polar coordinates are used, we write $dA = r\,dr\,d\theta$ and the triple integral is rewritten as an iterated integral in z, r, and θ:

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_1} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r\cos\theta,r\sin\theta)}^{h_2(r\cos\theta,r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta.$$

Fubini's Theorem still applies, allowing us to rearrange the order of integration.

Example 1: Find the volume of the solid bounded by the paraboloid $z = 36 - x^2 - y^2$ and the plane $z = 0$.



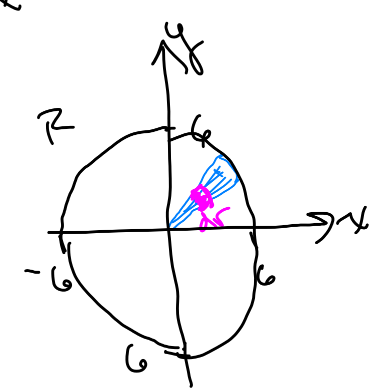
Trace in xy plane: $0 = 36 - x^2 - y^2$

$x^2 + y^2 = 36$ circle radius 6

Max value of z: 36

$$V = \iiint_E dV = \iint_R \left[\int_0^{36-x^2-y^2} 1 \, dz \right] dA$$

$$= \iint_R z \Big|_0^{36-x^2-y^2} dA = \iint_R (36 - x^2 - y^2)\, dA$$



$$= \int_0^{2\pi} \int_0^6 (36 - (\underbrace{x^2+y^2}_{r^2}))\, r\,dr\,d\theta$$

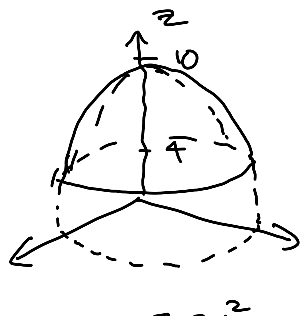
$dA = r\,dr\,d\theta$

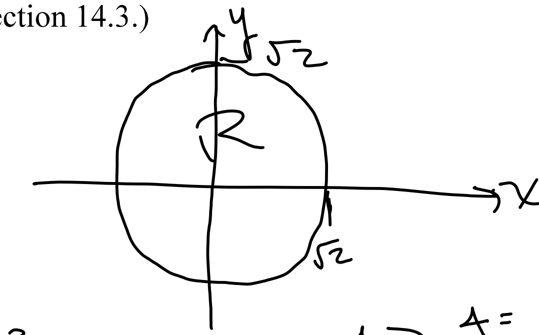
$$= \int_0^{2\pi} \int_0^6 (36 - r^2)\, r\,dr\,d\theta = \int_0^{2\pi} \int_0^6 (36r - r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{36r^2}{2} - \frac{r^4}{4} \right) \Big|_0^6 d\theta = \int_0^{2\pi} \left[18(6)^2 - \frac{6^4}{4} - 0 \right] d\theta = \int_0^{2\pi} 324\, d\theta$$

$$= 324\theta \Big|_0^{2\pi} = 324(2\pi - 0) = \boxed{648\pi}$$

Example 2: Find the volume of the solid bounded by the paraboloid $z = 10 - 3x^2 - 3y^2$ and the plane $z = 4$. (Same problem as Example 2 in Section 14.3.)





$$V = \iint_R \int_4^{10-3x^2-3y^2} 1 \, dz \, dA = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_4^{10-3x^2-3y^2} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}}\int_4^{10-3r^2} r \, dz \, dr \, d\theta$$

$z = 4 \Rightarrow 4 = 10 - 3x^2 - 3y^2$

$3x^2 + 3y^2 = 6$

$x^2 + y^2 = 2$

Circle radius $\sqrt{2}$

$\vdots$

$\boxed{6\pi}$ same as before

Example 3: Evaluate the iterated integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} dz \, dy \, dx$. (Hint: Rewrite using cylindrical coordinates.)

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} dz \, dy \, dx$$

$$= \int_0^{\pi/2}\int_0^3\int_0^{6-x-y} r \, dz \, dr \, d\theta$$

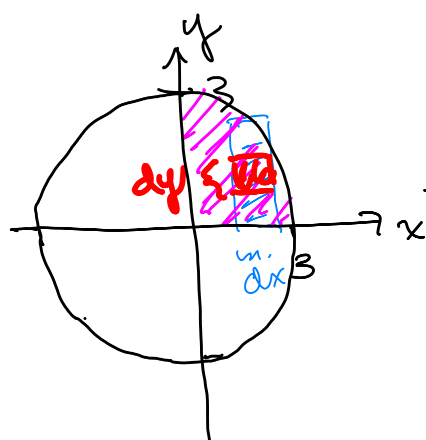
$$= \int_0^{\pi/2}\int_0^3\int_0^{6-r\cos\theta - r\sin\theta} r \, dz \, dr \, d\theta$$

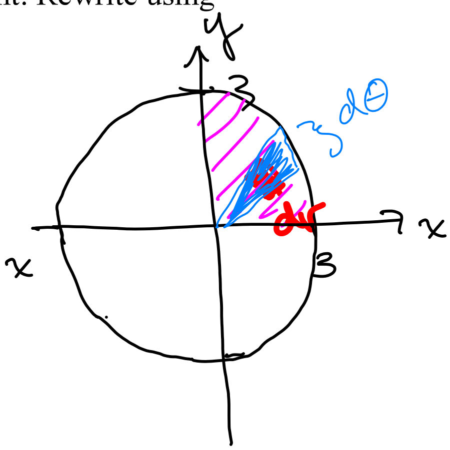
$$= \int_0^{\pi/2}\int_0^3 rz\Big|_0^{6-r\cos\theta - r\sin\theta} dr \, d\theta$$

$$= \int_0^{\pi/2}\int_0^3 r(6 - r\cos\theta - r\sin\theta) \, dr \, d\theta$$

$$= \int_0^{\pi/2}\int_0^3 [6r - r^2\cos\theta - r^2\sin\theta] \, dr \, d\theta$$

$$= \int_0^{\pi/2}\left[3r^2 - \frac{r^3}{3}(\cos\theta + \sin\theta)\right]\Big|_0^3 d\theta = \int_0^{\pi/2}\left[27 - 9(\cos\theta + \sin\theta)\right] d\theta$$





dy runs from $y = 0$ to $y = \sqrt{9 - x^2}$

$y^2 = 9 - x^2$

$y^2 + x^2 = 9$

circle of radius 3

dx runs from $x = 0$ to $x = 3$

(so Quadrant I only)

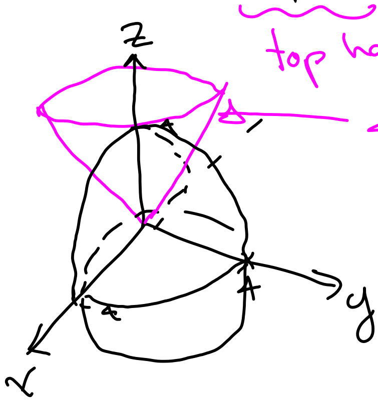
next page

$$= \left[27\theta - 9\sin\theta - 9(-\cos\theta)\right]\Big|_0^{\pi/2}$$

$$= 27\left(\frac{\pi}{2}\right) - 9\sin\frac{\pi}{2} + \cancel{9\cos\frac{\pi}{2}}^{0} - \cancel{27(0)}^{0} + \cancel{9\sin 0}^{0} - 9\cos 0 = \frac{27\pi}{2} - 9 + 0 - 9$$

$$\boxed{= \frac{27\pi}{2} - 18}$$

Example 4: Find the volume of solid inside the graph of $\underbrace{x^2 + y^2 + z^2 = 16}_{\text{sphere of radius 4}}$ and outside the graph of $\underbrace{z = \sqrt{x^2 + y^2}}_{\text{top half of cone}}$.

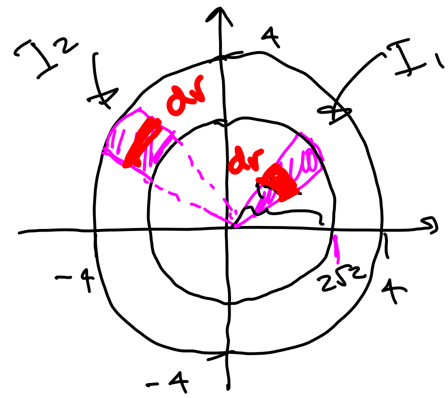


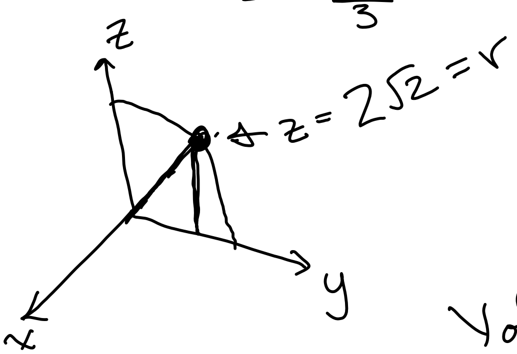
sides have ~~slope ±1~~ (sides are at 45° angle)

$$z = \sqrt{x^2+y^2}$$
$$z^2 = x^2+y^2$$
$$x = 0 \Rightarrow z^2 = y^2$$
$$z = \pm y$$
$$y = 0 \Rightarrow z^2 = x^2$$
$$z = \pm x$$
$$z = 0 \Rightarrow 0 = x^2+y^2 \Rightarrow x=0, y=0$$

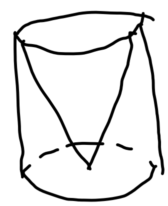
Bottom half of sphere:
use geometry formula.

$$V_{\frac{1}{2}\text{sphere}} = \frac{1}{2}\cdot\frac{4}{3}\pi R^3 = \frac{2}{3}\pi(4)^3 = \frac{128\pi}{3}$$





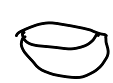
Volume



+

I_1

+



bottom half

Find the point of intersection between the cone and the sphere:

$x^2+y^2+z^2 = 16 \Rightarrow r^2 + z^2 = 16$ using $x^2+y^2 = r^2$

Cone: $z = \sqrt{x^2+y^2}$

$$z^2 = x^2+y^2$$

where they intersect:

Put z^2 into $\underbrace{x^2+y^2}_{z^2} + z^2 = 16$ in place of x^2+y^2

$$z^2 + z^2 = 16$$
$$2z^2 = 16$$
$$z^2 = 8$$
$$z = \pm 2\sqrt{2}$$

(Need 2 separate integrals I_1 and I_2)

See next page

Ex 4 cont'd:

$$I_1 = \int_0^{2\pi}\int_0^{2\sqrt{2}}\int_0^{r} r \, dz\, dr\, d\theta$$

$$= \frac{32\pi\sqrt{2}}{3}$$

roof of my solid: the cone

$$z = \sqrt{x^2+y^2}$$

$$z = \sqrt{r^2} = r$$

$$I_2 = \int_0^{2\pi}\int_{2\sqrt{2}}^{4}\int_0^{\sqrt{16-r^2}} r \, dz\, dr\, d\theta$$

$$= \frac{32\pi\sqrt{2}}{3}$$

eqn of sphere:

$$\underbrace{x^2+y^2}_{r^2}+z^2 = 16$$

$$r^2+z^2 = 16$$

$$z^2 = 16-r^2$$

$$z = \pm\sqrt{16-r^2}$$

Top half: $z=\sqrt{16-r^2}$

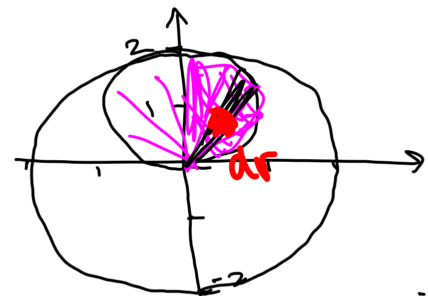
Volume $= I_1 + I_2 +$ Bottom half

$$= \frac{32\pi\sqrt{2}}{3} + \frac{32\pi\sqrt{2}}{3} + \frac{128\pi}{3} = \boxed{\frac{64\pi\sqrt{2}+128\pi}{3}}$$

Example 5: Find the volume of the solid cut from the sphere $x^2+y^2+z^2=4$ by the cylinder $r=2\sin\theta$.

↑ circle radius 1, center 1 unit from origin

sphere of radius 2



Find Volume of top half and double it.

$$V=2(2)\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

"Roof" of solid:

$$z^2=4-x^2-y^2$$

$$z=\pm\sqrt{4-x^2-y^2}$$

top half: $z=\sqrt{4-x^2-y^2}=\sqrt{4-r^2}$

$$=4\int_0^{\pi/2}\int_0^{2\sin\theta} rz\Big|_0^{\sqrt{4-r^2}}\,dr\,d\theta$$

$$=4\int_0^{\pi/2}\int_0^{2\sin\theta} r(4-r^2)^{1/2}\,dr\,d\theta$$

$u=4-r^2$
$du=-2r\,dr$
$-\frac{1}{2}du=r\,dr$

$$=4\left(-\frac{1}{2}\right)\int_0^{\pi/2}\frac{u^{3/2}}{3/2}\Big|_{r=0}^{r=2\sin\theta}\,d\theta$$

$$=-2\left(\frac{2}{3}\right)\int_0^{\pi/2}(4-r^2)^{3/2}\Big|_0^{2\sin\theta}\,d\theta$$

$$=-\frac{4}{3}\int_0^{\pi/2}\left[(4-4\sin^2\theta)^{3/2}-(4-0^2)^{3/2}\right]d\theta=-\frac{4}{3}\int_0^{\pi/2}\left[(4(1-\sin^2\theta))^{3/2}-4^{3/2}\right]d\theta$$

$$=-\frac{4}{3}\cdot 4^{3/2}\int_0^{\pi/2}\left[(\cos^2\theta)^{3/2}-1\right]d\theta=-\frac{4}{3}\cdot(4^{1/2})^3\int_0^{\pi/2}(\cos^3\theta-1)\,d\theta$$

$$=-\frac{4}{3}(2)^3\int_0^{\pi/2}\cos^3\theta\,d\theta+\frac{32}{3}\int_0^{\pi/2}1\,d\theta=-\frac{32}{3}\int_0^{\pi/2}\cos\theta\,(1-\sin^2\theta)\,d\theta+\frac{32}{3}\left(\frac{\pi}{2}-0\right)$$

$u=\sin\theta$
$du=\cos\theta\,d\theta$

$$=-\frac{32}{3}\int_0^{\pi/2}\cos\theta+\frac{32}{3}\int_0^{\pi/2}\cos\theta\sin^2\theta\,d\theta+\frac{16\pi}{3}$$

$$=-\frac{32}{3}\left(\sin\frac{\pi}{2}-\cancel{\sin 0}^{\,0}\right)+\frac{32}{3}\cdot\frac{\sin^3\theta}{3}\Big|_0^{\pi/2}+\frac{16\pi}{3}$$

$$=-\frac{32}{3}(1)+\frac{32}{9}\left[\left(\sin\frac{\pi}{2}\right)^3-\cancel{(\sin 0)^3}^{\,0}\right]+\frac{16\pi}{3}=-\frac{32}{3}+\frac{32}{9}+\frac{16\pi}{3}=-\frac{96}{9}+\frac{32}{9}+\frac{16\pi}{3}$$

$$=-\frac{64}{9}+\frac{16\pi}{3}=-\frac{64}{9}+\frac{48\pi}{9}=\boxed{\frac{48\pi-64}{9}}=\boxed{\frac{16(3\pi-4)}{9}}$$

Previous example: Volume = $\boxed{\frac{16(3\pi-4)}{9}}$

Spherical coordinates:

Recall: In spherical coordinates, we write point *P* as (ρ,θ,φ), where ρ is the distance from the origin *O* to *P*, θ is the same angle used in cylindrical coordinates (for $r \ge 0$), and φ is the angle between $\overrightarrow{OP}$ and the positive *z*-axis.

Equations relating spherical coordinates to rectangular and cylindrical coordinates:

$x = \rho \sin\phi \cos\theta$ $\qquad$ $y = \rho \sin\phi \sin\theta$ $\qquad$ $z = \rho\cos\phi$

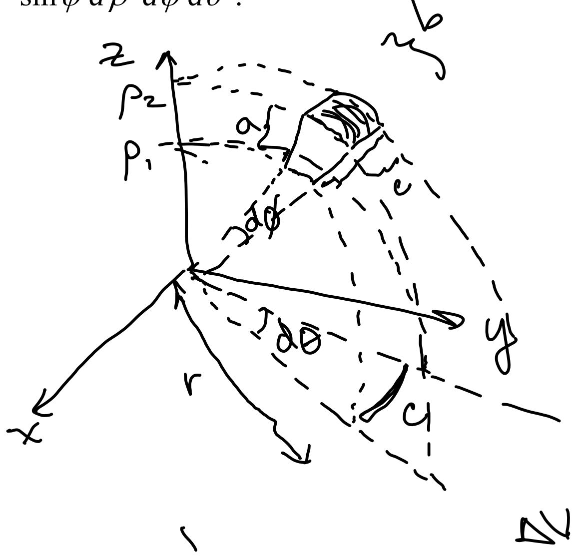
$x^2 + y^2 + z^2 = \rho^2$ $\qquad$ $\tan\theta = \dfrac{y}{x}$ $\qquad$ $\phi = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$

$r^2 = \rho^2\sin^2\phi$ $\qquad$ $\rho = \sqrt{r^2+z^2}$

↳ so $r = \rho\sin\phi$ if everything is positive

If we start with $\iiint_Q f(x,y,z)\,dV$, the volume increment dV can be written as

$\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.



Block $dV \approx abc$

$a = \rho_2 - \rho_1 = \Delta\rho$

For the others, remember that arc length = (radius)(central angle)

$b = \rho\,\Delta\phi$

$c = r\,\Delta\theta = \rho\sin\phi\,\Delta\theta$

$\Delta V = abc = \Delta\rho(\rho\Delta\phi)(\rho\sin\phi\,\Delta\theta) = \rho^2\sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$

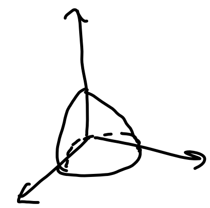
So, our triple integral becomes

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_1}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(x,y,z)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \ .$$

So, $dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

Example 6: Find the volume of the sphere $x^2 + y^2 + z^2 = R^2$ using spherical coordinates.

Find 1st octant volume and multiply by 8:



$$V = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{R} 1\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^3}{3}\Big|_0^{R} \sin\phi\, d\phi\, d\theta = 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{R^3}{3}\sin\phi\, d\phi\, d\theta$$

$$= \frac{8R^3}{3}\int_0^{\pi/2} (-\cos\phi)\Big|_0^{\pi/2} d\theta = \frac{8R^3}{3}\int_0^{\pi/2}\left(-\cancelto{0}{\cos\tfrac{\pi}{2}} + \cos 0\right) d\theta$$

$$= \frac{8R^3}{3}\int_0^{\pi/2} 1\, d\theta$$

$$= \frac{8R^3}{3}\,\theta\Big|_0^{\pi/2}$$

$$= \frac{8R^3}{3}\left(\frac{\pi}{2} - 0\right)$$

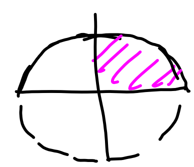
$$= \frac{8\pi R^3}{6} = \frac{4}{3}\pi R^3$$

Example 7: Rewrite the iterated integral $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\; dz\, dy\, dx$ in both cylindrical and spherical coordinates. Choose the easier form to integrate.

$$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} \sqrt{\underbrace{x^2+y^2}_{r^2}+z^2}\; dz\, dy\, dx$$

Cylindrical:

$$\int_0^{\pi/2}\int_0^{3}\int_0^{\sqrt{9-r^2}} r\sqrt{r^2+z^2}\; dz\, dr\, d\theta$$



$y = \sqrt{9-x^2}$

$y^2 = 9 - x^2$

$x^2 + y^2 = 3$

$dA = r\,dr\,d\theta$

Spherical:

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} \rho\cdot\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

much easier to integrate!

Should get $\boxed{\frac{81\pi}{8}}$

$\rho^2 = x^2 + y^2 + z^2$

$\rho = \sqrt{x^2+y^2+z^2}$

$dz\,dy\,dx = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

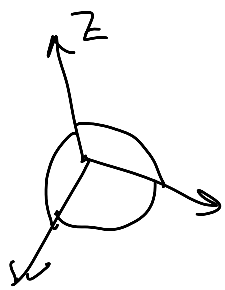
$0 \le z \le \sqrt{9-x^2-y^2}$

$z = \sqrt{9-x^2-y^2}$

$z^2 = 9 - x^2 - y^2$

$z^2 + x^2 + y^2 = 9$ sphere

So we have 1st octant of this sphere



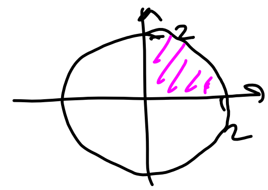
(we didn't do this one in class)

Example 8: Rewrite the iterated integral $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \underbrace{\sqrt{x^2+y^2}}_{r}\, dz\, dy\, dx$ in both cylindrical and spherical coordinates.

y runs from $y=0$ to $y=\sqrt{4-x^2}$

$y^2=4-x^2$

$x^2+y^2=4$



Cylindrical: $\displaystyle\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{16-r^2}} r\cdot r\, dz\, dr\, d\theta$

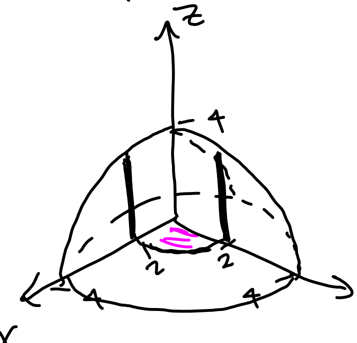
Spherical: (much more difficult to set up!)

$\sqrt{x^2+y^2}=r=\rho\sin\phi$

$dV=\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$\displaystyle\int\int\int (\rho\sin\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

To get limits of integration, $\boxed{\text{See next page}}$



Note

dz goes from $z=0$ to $z=\sqrt{16-x^2-y^2}$

$z^2=16-x^2-y^2$

$x^2+y^2+z^2=16$

sphere radius 4

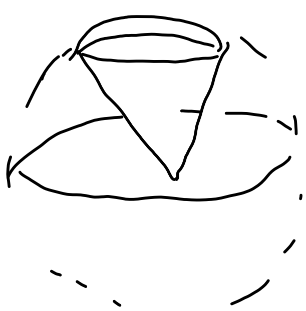
Sphere has radius 4, cylinder has radius 2. So this solid is a cylinder with a spherical "roof"

dy goes from $y=0$ to $y=\sqrt{4-x^2}$

$y^2=4-x^2$

$x^2+y^2=4$ cylinder radius 2

Example 9: Find the volume of the solid bounded below by the upper half of the cone $z^2=x^2+y^2$ and above by the sphere $x^2+y^2+z^2=9$.



In spherical coordinates, eqn of sphere is $\rho=3$

Where do sphere and cone intersect?

$\underbrace{x^2+y^2}+z^2=9$

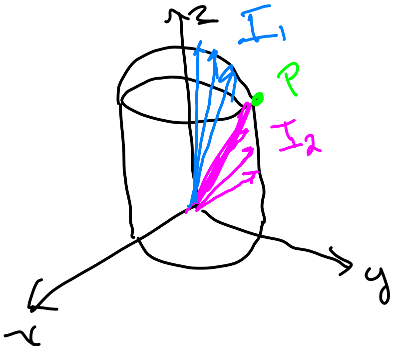
z^2 on cone $x^2+y^2=z^2$

$z^2+z^2=9 \Rightarrow 2z^2=9 \Rightarrow z^2=\frac{9}{2}$

$z=\pm\frac{3}{\sqrt{2}}$

$\boxed{\text{See the page after the next page. :)}}$

Ex 8 cont'd:



To figure out the limits, note that the rays emerging from the origin are bounded by the cylinder on the sides (I_1), and by the spherical cap near the top (I_2).

We must split into 2 integrals.

Must also find where sphere & cylinder intersect.

cylinder: $y = \sqrt{4-x^2}$
$x^2+y^2=4$

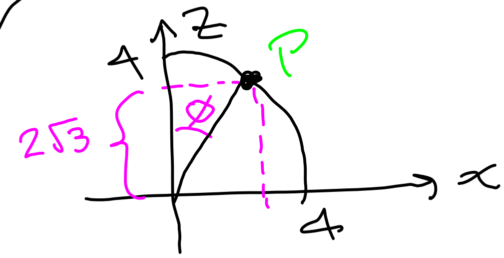
sphere: $z = \sqrt{16-x^2-y^2}$
$x^2+y^2+z^2=16 \Rightarrow x^2+y^2 = 16-z^2$

set x^2+y^2's equal: $4 = 16 - z^2$

$z^2 = 12$

$z = \pm\sqrt{12} = \pm 2\sqrt{3}$

we want $z = 2\sqrt{3}$



$4\cos\phi = 2\sqrt{3}$

$\cos\phi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

$\phi = \frac{\pi}{6}$

Integral in spherical coordinates is $I_1 + I_2$, where

$$I_1: \int_0^{\pi/2}\int_0^{\pi/6}\int_0^4 \rho^3 \sin^2\phi \, d\rho \, d\phi \, d\theta$$

(remember, we only want the 1st octant portion, because of the original bounds on x and y.)

$$I_2: \int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\int_0^{\csc\phi} \rho^3 \sin^2\phi \, d\rho \, d\phi \, d\theta$$

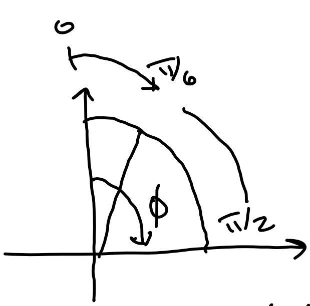
So, for I_1, $d\phi$ goes from 0 to $\frac{\pi}{6}$.

For I_2, $d\phi$ goes from $\frac{\pi}{6}$ to $\frac{\pi}{2}$.

For I_1, $d\rho$ goes from 0 to 4. (bounded by spherical cap)

For I_2, $d\rho$ goes from 0 to ? It goes until it hits the cylinder.

Need to write that ρ in terms of ϕ or θ.



convert $x^2+y^2=4$ to spherical:

$(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = 4$

$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = 4$

$\rho^2\sin^2\phi\,(\underbrace{\cos^2\theta+\sin^2\theta}_{1}) = 4$

$\rho^2\sin^2\phi = 4$

$\rho\sin\phi = 2$

$\rho = \frac{2}{\sin\phi} = 2\csc\phi$

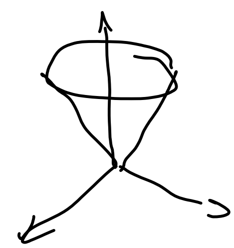
Ex 9 cont'd:

Also, $z = \rho\cos\phi$, so then, at this point

we have $\rho = 3$, $z = \frac{3}{\sqrt{2}}$

$$z = \rho\cos\phi$$
$$\frac{3}{\sqrt{2}} = 3\cos\phi$$
$$\frac{1}{\sqrt{2}} = \cos\phi$$
$$\cos\phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} \rho^2 \sin\phi \, d\phi \, d\theta$$



work it out .... should get $\boxed{9\pi(2-\sqrt{2})}$

Details: $V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} \rho^2 \sin\phi \, d\phi \, d\theta$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\Big|_0^3 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \left(\frac{3^3}{3} - \frac{0^3}{3}\right)\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} 9 \sin\phi \, d\phi \, d\theta = 9\int_0^{2\pi} (-\cos\phi)\Big|_0^{\pi/4} d\theta$$

$$= 9\int_0^{2\pi} \left(-\cos\frac{\pi}{4} + \cos 0\right) d\theta = 9\int_0^{2\pi} \left(-\frac{\sqrt{2}}{2} + 1\right) d\theta$$

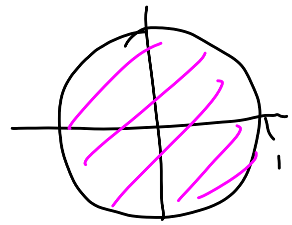
$$= 9\left(-\frac{\sqrt{2}}{2} + 1\right)\theta\Big|_0^{2\pi} = 9\left(-\frac{\sqrt{2}}{2} + 1\right)(2\pi - 0) = -\frac{18\pi\sqrt{2}}{2} + 18\pi$$

$$= \boxed{-9\pi\sqrt{2} + 18\pi} = \boxed{9\pi(2-\sqrt{2})}$$

Example 10: Set up integrals in cylindrical and spherical coordinates to evaluate $\iiint_H (x^2+y^2)\,dV$, where H is the hemispherical region that lies above the *xy*-plane and below the sphere $x^2+y^2+z^2=1$.

Cylindrical: $x^2+y^2=r^2$, $dV=r\,dr\,d\theta$

$$\iiint_H (x^2+y^2)\,dV=\int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{1-r^2}} r^2\cdot r\,dz\,dr\,d\theta$$



$z^2=1-x^2-y^2$

$z^2=1-(x^2+y^2)$

$z^2=1-r^2$

$z=\sqrt{1-r^2}$

Spherical: $x^2+y^2=(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2$

$=\rho^2\sin^2\phi\,(\underbrace{\cos^2\theta+\sin^2\theta}_{1})=\rho^2\sin^2\phi$

$dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$$\iiint_H (x^2+y^2)\,dV=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{1}\left(\rho^2\sin^2\phi\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Both integrals will give a result of $\frac{4\pi}{15}$.