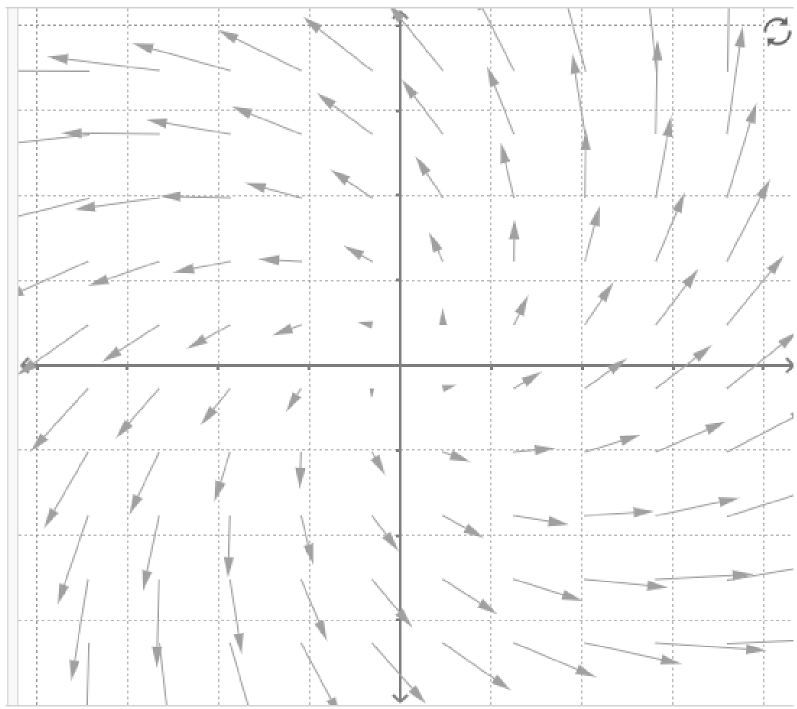
15.1: Vector Fields

Vector fields are functions that assign a vector to each point in $\mathbb{R}^2$ or $\mathbb{R}^3$.

We can represent a vector field by a mesh of arrows at regularly spaced intervals.

Example 1: This is a graphic representation of the vector field $\mathbf{F}(x, y) = \langle x - y, x + y \rangle$.



Example 2: Plot a few vectors in the field for $\mathbf{F}(x, y) = \langle x^2 y, xy^2 \rangle$.

Note: A gradient function $\nabla f(x,y) = \left\langle f_x(x,y),\, f_y(x,y) \right\rangle$, or $\nabla f(x,y,z) = \left\langle f_x(x,y,z),\, f_y(x,y,z),\, f_z(x,y,z) \right\rangle$, is a vector field. Physical examples of vector fields include velocity fields, gravitational fields, and electric force fields.

Conservative vector fields and potential functions:

Definition: A vector field **F** is called *conservative* if there exists a differentiable function f such that $\mathbf{F} = \nabla f$. The function f is called the *potential function* for **F**.

Example 3: Find a conservative vector field for the potential function $f(x,y) = x^3y^2 + 2xy^4$.

Theorem: Test for a Conservative Vector Field (in $\mathbb{R}^2$)

Let functions M and N have continuous first partial derivatives on an open disk R. The vector field $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Example 4: Determine whether the vector field $\mathbf{F}(x, y) = (x^2 y^3 - 2x)\mathbf{i} + (3x^4 y - 2y)\mathbf{j}$ is conservative. If it is, find a potential function for the vector field.

Example 5: Determine whether the vector field $\mathbf{F}(x, y) = (x^3 y^2 - 4x)\mathbf{i} + (\frac{1}{2}x^4 y - 6y)\mathbf{j}$ is conservative. If it is, find a potential function for the vector field.

Example 6: Determine whether the vector field $\mathbf{F}(x, y) = 3x^2 y^2\mathbf{i} + 2x^3 y\mathbf{j}$ is conservative. If it is, find a potential function for the vector field.

Curl of a vector field:

<u>Definition</u>: <u>Curl</u>

The *curl* of $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\begin{aligned}\operatorname{curl}\mathbf{F}(x,y,z) &= \nabla \times \mathbf{F}(x,y,z) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.\end{aligned}$$

If $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is said to be *irrotational*.

<u>Example 7:</u> Find the curl of $\mathbf{F}(x,y,z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$ at the point $(2,-1,3)$.

<u>Theorem</u>: <u>Test for a Conservative Vector Field</u> (in $\mathbb{R}^3$)

Let functions M, N, and P have continuous first partial derivatives on an open sphere Q in $\mathbb{R}^3$. The vector field $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if

$$\operatorname{curl}\mathbf{F}(x,y,z) = \mathbf{0}.$$

Consequently, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\ \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z},\ \text{and}\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Example 8: Determine whether the vector field $\mathbf{F}(x,y,z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$ is conservative. If it is, find a potential function for the vector field.

Example 9: Determine whether the vector field $\mathbf{F}(x,y,z) = xyz\mathbf{i} - y^2\mathbf{j} + xz\mathbf{k}$ is conservative. If it is, find a potential function for the vector field.

Example 10: Determine whether the vector field $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$ is conservative. If it is, find a potential function for the vector field.

Divergence of a vector field:

Definition: Divergence

The *divergence* of $\mathbf{F}(x,y)=M\mathbf{i}+N\mathbf{j}$ is

$$\operatorname{div}\mathbf{F}(x,y)=\nabla\cdot\mathbf{F}(x,y)=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}.$$

The *divergence* of $\mathbf{F}(x,y,z)=M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ is

$$\operatorname{div}\mathbf{F}(x,y,z)=\nabla\cdot\mathbf{F}(x,y,z)=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}.$$

If $\operatorname{div}\mathbf{F}=0$, then $\mathbf{F}$ is said to be *divergence free*.

Note: $\operatorname{curl}\mathbf{F}$ is a vector function; $\operatorname{div}\mathbf{F}$ is a scalar function. Because $\operatorname{curl}\mathbf{F}$ is defined as a cross product, it does not make sense in $\mathbb{R}^2$.

Example 11: Find the divergence of $\mathbf{F}(x,y,z)=\ln(xyz)(\mathbf{i}+\mathbf{j}+\mathbf{k})$ at the point $(3,2,1)$.

Example 12: Given $\mathbf{F}(x,y,z)=e^{3x}y^2\mathbf{i}+2xy^3z^4\mathbf{j}+y^3\sin(2z)\mathbf{k}$, find (a) $\operatorname{div}\mathbf{F}(x,y,z)$, (b) $\operatorname{div}\mathbf{F}(0,2,0)$, and (c) $\operatorname{div}(\operatorname{curl}\mathbf{F}(x,y,z))$.

Theorem:

If $\mathbf{F}(x,y,z)=M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ is a vector field and M, N, and P have continuous second partial derivatives, then

$$\operatorname{div}(\operatorname{curl}\mathbf{F}(x,y,z))=0.$$