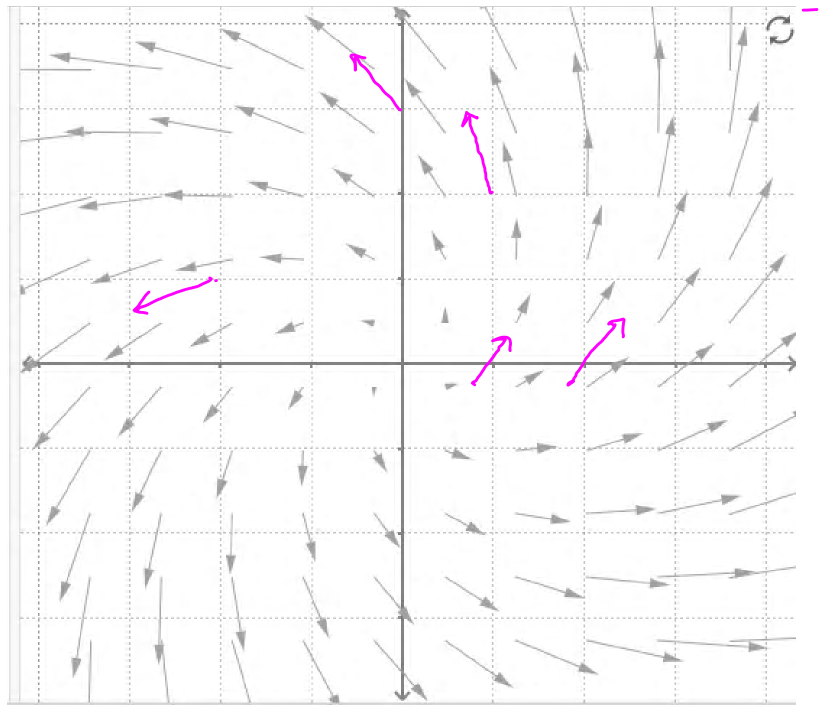
15.1: Vector Fields

Vector fields are functions that assign a vector to each point in $\mathbb{R}^2$ or $\mathbb{R}^3$.

We can represent a vector field by a mesh of arrows at regularly spaced intervals.

Example 1: This is a graphic representation of the vector field $\mathbf{F}(x,y) = \langle x-y, x+y \rangle$.

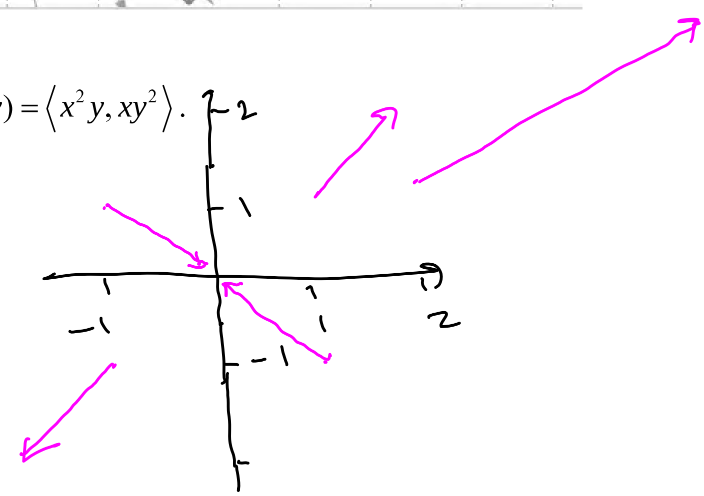


Plot a few vectors:

$\vec{F}(x,y) = \langle x-y, x+y \rangle$

(x,y)	$\vec{F}(x,y)$
$(0,0)$	$\langle 0,0 \rangle$
$(1,0)$	$\langle 1-0, 1+0 \rangle = \langle 1,1 \rangle$
$(2,0)$	$\langle 2-0, 2+0 \rangle = \langle 2,2 \rangle$
$(1,2)$	$\langle 1-2, 1+2 \rangle = \langle -1,3 \rangle$
$(0,3)$	$\langle 0-3, 0+3 \rangle = \langle -3,3 \rangle$
$(-2,1)$	$\langle -2-1, -2+1 \rangle = \langle -3,-1 \rangle$

Example 2: Plot a few vectors in the field for $\mathbf{F}(x,y) = \langle x^2 y, xy^2 \rangle$.



$\vec{F}(x,y) = \langle x^2y, xy^2 \rangle$

$\vec{F}(0,0) = \langle 0,0 \rangle$

$\vec{F}(1,0) = \langle 0,0 \rangle$

$\vec{F}(x,y) = \langle 0,0 \rangle$ whenever (x,y) is on either axis

$\vec{F}(1,1) = \langle 1,1 \rangle$

$\vec{F}(-1,1) = \langle (-1)^2(1), -1(1)^2 \rangle = \langle 1,-1 \rangle$

$F(1,-1) = \langle -1, 1 \rangle$

$F(-1,-1) = \langle -1, -1 \rangle$

$F(2,1) = \langle 2^2(1), 2(1)^2 \rangle = \langle 4,2 \rangle$

Note: A gradient function $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$, or $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$, is a vector field. Physical examples of vector fields include velocity fields, gravitational fields, and electric force fields.

Conservative vector fields and potential functions:

Definition: A vector field **F** is called *conservative* if there exists a differentiable function *f* such that $\mathbf{F} = \nabla f$. The function *f* is called the *potential function* for **F**.

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$

Example 3: Find a conservative vector field for the potential function $f(x,y) = x^3y^2 + 2xy^4$.

Conservative vector field is $\vec{F}(x,y) = \nabla f(x,y)$

$= \langle f_x(x,y), f_y(x,y) \rangle$

$= \langle 3x^2y^2 + 2y^4, 2x^3y + 8xy^3 \rangle$

$$\boxed{\vec{F}(x,y) = (3x^2y^2 + 2y^4)\vec{i} + (2x^3y + 8xy^3)\vec{j}}$$

Theorem: Test for a Conservative Vector Field (in $\mathbb{R}^2$)

Let functions *M* and *N* have continuous first partial derivatives on an open disk *R*. The vector field $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Why is this true?

Suppose $\vec{F} = \nabla f$, where $\vec{F}(x,y) = \langle M, N \rangle$ and M, N have continuous 1st partials

$\nabla f(x,y) = \langle M, N \rangle$

Then $f_x = M$ and $f_y = N$.

So $f_{xy} = \frac{\partial M}{\partial y}$ and $f_{yx} = \frac{\partial N}{\partial x}$.

Because the 1st partials are continuous, the mixed second partials f_{xy} and f_{yx} must be equal.

So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example 4: Determine whether the vector field $\mathbf{F}(x,y) = (x^2y^3 - 2x)\mathbf{i} + (3x^4y - 2y)\mathbf{j}$ is conservative. If it is, find a potential function for the vector field.

$M = x^2y^3 - 2x$

$N = 3x^4y - 2y$

$\dfrac{\partial M}{\partial y} = 3x^2y^2$ $\quad \dfrac{\partial N}{\partial x} = 12x^3y$. No, not a conservative vector field.

(because $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$)

Example 5: Determine whether the vector field $\mathbf{F}(x,y) = (x^3y^2 - 4x)\mathbf{i} + (\frac{1}{2}x^4y - 6y)\mathbf{j}$ is conservative. If it is, find a potential function for the vector field.

$M = x^3y^2 - 4x$, $N = \frac{1}{2}x^4y - 6y$

$\dfrac{\partial M}{\partial y} = 2x^3y$, $\dfrac{\partial N}{\partial x} = \frac{1}{2}(4x^3)y = 2x^3y \Rightarrow \vec{F}$ is conservative.

Now find potential function f such that $\vec{F} = \nabla f$.

$\nabla f(x,y) = \langle M, N \rangle$

$f_x = M \Rightarrow f(x,y) = \int f_x(x,y)\,dx = \int M\,dx$

$f(x,y) = \int M\,dx = \int (x^3y^2 - 4x)\,dx = \frac{1}{4}x^4y^2 - 2x^2 + g(y)$

These 2 expressions for f must be equal.

Let $g(y) = -3y^2$, $h(x) = -2x^2$

$\boxed{f(x,y) = \frac{1}{4}x^4y^2 - 2x^2 - 3y^2 + K}$

$f_y = N \Rightarrow f(x,y) = \int f_y(x,y)\,dy = \int N\,dy$

$f(x,y) = \int N\,dx = \int (\frac{1}{2}x^4y - 6y)\,dy$

$= \frac{1}{2}x^4 \cdot \frac{y^2}{2} - \frac{6y^2}{2} + h(x)$

$= \frac{1}{4}x^4y^2 - 3y^2 + h(x)$

(Check that $\nabla f = \vec{F}$)

Example 6: Determine whether the vector field $\mathbf{F}(x,y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$ is conservative. If it is, find a potential function for the vector field.

Is it conservative? $M = 3x^2y^2$, $N = 2x^3y$

$\dfrac{\partial M}{\partial y} = 6x^2y$, $\dfrac{\partial N}{\partial x} = 6x^2y$. Yes, $\vec{F}$ is conservative.

$f(x,y) = \int f_x(x,y)\,dx = \int M\,dx = \int 3x^2y^2\,dx = 3\frac{x^3}{3}y^2 + g(y) = x^3y^2 + g(y)$

$f(x,y) = \int f_y(x,y)\,dy = \int N\,dy = \int 2x^3y\,dy = 2x^3\frac{y^2}{2} + h(x) = x^3y^2 + h(x)$

Let $g(y) = h(x) = K$: $\boxed{f(x,y) = x^3y^2 + K}$

Check: $\nabla f(x,y) = \langle 3x^2y^2, 2x^3y \rangle = \vec{F}(x,y)$ ✓OK

Curl of a vector field:

Definition: Curl

The *curl* of $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\begin{aligned}\operatorname{curl}\mathbf{F}(x,y,z) &= \nabla \times \mathbf{F}(x,y,z) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.\end{aligned}$$

If $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is said to be *irrotational*.

Example 7: Find the curl of $\mathbf{F}(x,y,z) = x^2 z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$ at the point $(2,-1,3)$.

$$\operatorname{curl}\vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 z & -2xz & yz \end{vmatrix} = \langle z-(-2x), -(0-x^2), -2z-0\rangle$$

$$= \langle z+2x, x^2, -2z\rangle$$

$$\operatorname{curl}\vec{F}(2,-1,3) = \langle 3+2(2), 2^2, -2(3)\rangle$$

$$= \boxed{\langle 7, 4, -6\rangle}$$

Theorem: Test for a Conservative Vector Field (in $\mathbb{R}^3$)

Let functions M, N, and P have continuous first partial derivatives on an open sphere Q in $\mathbb{R}^3$. The vector field $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if

$$\operatorname{curl}\mathbf{F}(x,y,z) = \mathbf{0}.$$

Consequently, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}, \text{ and } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Example 8: Determine whether the vector field $\mathbf{F}(x,y,z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$ is conservative. If it is, find a potential function for the vector field.

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \langle 6xyz^2 - 6xyz^2,\ 3y^2z^2 - 3y^2z^2,\ 2yz^3 - 2yz^3 \rangle$$

$= \langle 0,0,0 \rangle = \vec{0}$. So $\vec{F}$ is conservative.

$f(x,y,z) = \int M\,dx = \int y^2z^3\,dx = xy^2z^3 + g(y,z)$

$f(x,y,z) = \int N\,dy = \int 2xyz^3\,dy = xy^2z^3 + h(x,z)$

$f(x,y,z) = \int P\,dz = \int 3xy^2z^2\,dz = xy^2z^3 + P(x,y)$

Let $g(y,z) = h(x,z) = P(x,y) = K$

$\boxed{f(x,y,z) = xy^2z^3 + K}$

Example 9: Determine whether the vector field $\mathbf{F}(x,y,z) = xyz\mathbf{i} - y^2\mathbf{j} + xz\mathbf{k}$ is conservative. If it is, find a potential function for the vector field.

$$\text{curl } \vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & -y^2 & xz \end{vmatrix} = \langle 0-0,\ xy - z,\ 0 - xz \rangle$$

$= \langle 0,\ xy - z,\ -xz \rangle$

$\boxed{\text{No, } \vec{F} \text{ is } \underline{\text{not}} \text{ conservative.}}$

Example 10: Determine whether the vector field $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (x^2+z^2)\mathbf{j} + 2yz\mathbf{k}$ is conservative. If it is, find a potential function for the vector field.

$$\text{curl } \vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2+z^2 & 2yz \end{vmatrix} = \langle 2z - 2z,\ 0 - 0,\ 2x - 2x \rangle$$

$= \langle 0,0,0 \rangle$. Yes, $\vec{F}$ is conservative.

$f(x,y,z) = \int f_x\,dx = \int 2xy\,dx = \frac{2x^2}{2}y + g(y,z) = x^2y + g(y,z)$

Now differentiate $f(x,y,z) = x^2y + g(y,z)$ with respect to y:

$f_y(x,y,z) = x^2 + \frac{\partial g}{\partial y}(y,z)$ ← must be equal to $x^2 + z^2 = N = f_y$

Set $f_y = f_y$: $\quad x^2 + \frac{\partial g}{\partial y}(y,z) = x^2 + z^2$

Therefore $\frac{\partial g}{\partial y}(y,z) = z^2$

$g(y,z) = \int \frac{\partial g}{\partial y}(y,z)\,dy = \int z^2\,dy = z^2y + h(z)$

$f(x,y,z) = x^2y + g(y,z) = x^2y + z^2y + h(z)$. Need to find $h(z)$.

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$f(x,y,z) = x^2y + z^2y + h(z)$

$f_z(x,y,z) = 2zy + h'(z)$. Set equal to $f_z = P = 2yz$

$2zy + h'(z) = 2yz \Rightarrow h'(z) = 0 \Rightarrow h(z) = K$

Divergence of a vector field: $\boxed{f(x,y,z) = x^2y + z^2y + K}$

Definition: Divergence

The *divergence* of $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is

$$\operatorname{div}\mathbf{F}(x,y) = \nabla\cdot\mathbf{F}(x,y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

$\operatorname{div}\vec{F}(x,y) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \cdot \langle M, N \rangle$

$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

The *divergence* of $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\operatorname{div}\mathbf{F}(x,y,z) = \nabla\cdot\mathbf{F}(x,y,z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

If $\operatorname{div}\mathbf{F} = 0$, then $\mathbf{F}$ is said to be *divergence free*.

$\operatorname{div}\vec{F}(x,y) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle M, N, P \rangle$

$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

Note: $\operatorname{curl}\mathbf{F}$ is a vector function; $\operatorname{div}\mathbf{F}$ is a scalar function. Because $\operatorname{curl}\mathbf{F}$ is defined as a cross product, it does not make sense in $\mathbb{R}^2$.

Example 11: Find the divergence of $\mathbf{F}(x,y,z) = \ln(xyz)(\mathbf{i}+\mathbf{j}+\mathbf{k})$ at the point $(3,2,1)$.

$F(x,y,z) = \langle \ln(xyz), \ln(xyz), \ln(xyz) \rangle$

$\operatorname{div} F(x,y,z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \frac{1}{xyz}(yz) + \frac{1}{xyz}(xz) + \frac{1}{xyz}(xy)$

$= \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$

$\operatorname{div}(3,2,1) = \frac{1}{3} + \frac{1}{2} + \frac{1}{1}$

$= \frac{2}{6} + \frac{3}{6} + \frac{6}{6}$

$= \boxed{\frac{11}{6}}$

Example 12: Given $\mathbf{F}(x,y,z) = e^{3x}y^2\mathbf{i} + 2xy^3z^4\mathbf{j} + y^3\sin(2z)\mathbf{k}$, find (a) $\operatorname{div}\mathbf{F}(x,y,z)$, (b) $\operatorname{div}\mathbf{F}(0,2,0)$, and (c) $\operatorname{div}(\operatorname{curl}\mathbf{F}(x,y,z))$.

(a) $\operatorname{div} F(x,y,z) = 3e^{3x}y^2 + 6xy^2z^4 + 2y^3\cos(2z)$

(b) $\operatorname{div} F(0,2,0) = 3e^0(2)^2 + 6(0) + 2(2)^3\cos 0 = 12 + 16 = \boxed{28}$

(c) $\operatorname{curl}\vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{3x}y^2 & 2xy^3z^4 & y^3\sin 2z \end{vmatrix}$

$= \langle 3y^2\sin 2z - 8xy^3z^3, 0 - 0, 2y^3z^4 - 2e^{3x}y \rangle$

Theorem:

If $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field and M, N, and P have continuous second partial derivatives, then

$$\operatorname{div}(\operatorname{curl}\mathbf{F}(x,y,z)) = 0.$$

$\operatorname{div}(\operatorname{curl}\vec{F}(x,y,z)) = -8y^3z^3 + 8y^3z^3 = \boxed{0}$