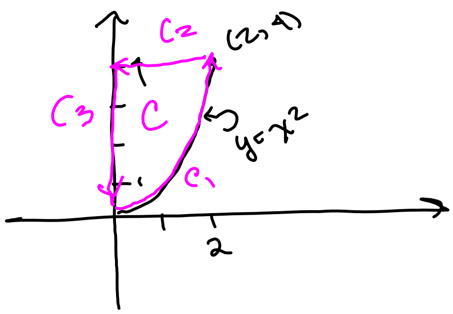
15.2: Line Integrals

Recall: A curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is considered *smooth* on the interval (a,b) if $x'(t)$, $y'(t)$, and $z'(t)$ are continuous on (a,b), and if $\mathbf{r}'(t) \neq \mathbf{0}$ for every t in (a,b). (In other words, if the curve is to be smooth, then $x'(t)$, $y'(t)$, and $z'(t)$ cannot be simultaneously 0 anywhere in the interval.)

Definition: A curve C is *piecewise smooth* on $[a,b]$ if the $[a,b]$ can be partitioned into a finite number of subintervals on which C is smooth.

Example 1: Find a piecewise smooth parametrization of the path shown.



C_1: $\vec{r}_1(t) = \langle t, t^2 \rangle$, $0 \le t \le 2$

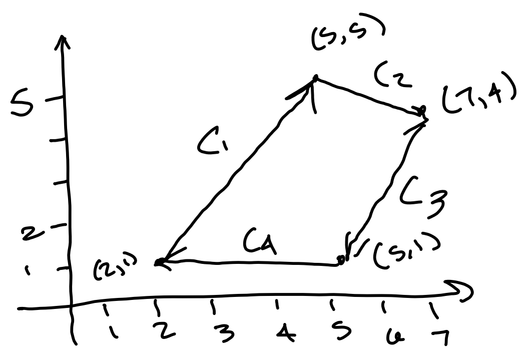
C_2: $\vec{r}_2(t) = \langle -t+4, 4 \rangle$, $2 \le t \le 4$

C_3: $\vec{r}_3(t) = \langle 0, -t+8 \rangle$, $4 \le t \le 8$

$$\vec{r}(t) = \begin{cases} \langle t, t^2 \rangle, & \text{if } 0 \le t \le 2 \\ \langle -t+4, 4 \rangle, & \text{if } 2 \le t \le 4 \\ \langle 0, -t+8 \rangle, & \text{if } 4 \le t \le 8 \end{cases}$$

so $\vec{r}(t)$ is def

Example 2: Find a piecewise smooth parametrization of the path shown.



C_1: $\vec{u} = \vec{PQ} = \langle 5-2, 5-1 \rangle = \langle 3, 4 \rangle$

$\vec{r}_1(t) = \langle 2, 1 \rangle + t\langle 3, 4 \rangle = \langle 2+3t, 1+4t \rangle$, $0 \le t \le 1$

C_2: $\vec{u} = \langle 2, -1 \rangle$

$\vec{r}_2(t) = \langle 5, 5 \rangle + (t-1)\langle 2, -1 \rangle$, $1 \le t \le 2$

$= \langle 5+2t-2, 5-t+1 \rangle$

$= \langle 3+2t, 6-t \rangle$

C_3: $\vec{u} = \langle -2, -3 \rangle$, $2 \le t \le 3$

$\vec{r}_3(t) = \langle 7, 4 \rangle + (t-2)\langle -2, -3 \rangle = \langle 11-2t, 10-3t \rangle$

$= \langle 7-2t+4, 4-3t+6 \rangle$

C_4: $\vec{u} = \langle -3, 0 \rangle$, $3 \le t \le 4$

$\vec{r}_4(t) = \langle 5, 1 \rangle + (t-3)\langle -3, 0 \rangle$

$= \langle 5-3t+9, 1 \rangle = \langle 14-3t, 1 \rangle$

$$\boxed{\vec{r}(t) = \begin{cases} \langle 2+3t, 1+4t \rangle, & \text{if } 0 \le t \le 1 \\ \langle 3+2t, 6-t \rangle, & \text{if } 1 \le t \le 2 \\ \langle 11-2t, 10-3t \rangle, & \text{if } 2 \le t \le 3 \\ \langle 14-3t, 1 \rangle, & \text{if } 3 \le t \le 4 \end{cases}}$$

Line integrals:

Definition: Suppose f is defined in a region containing a smooth curve C of finite length. Suppose also that C is partitioned into n subarcs, with Δs_i representing the length of the ith subarc, and with $\|\Delta\|$ representing the length of the longest subarc. Then the line integral of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{\|\Delta\|\to 0, n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i .$$

Theorem: Evaluating a Line Integral

Suppose f is continuous in a region containing a smooth curve C. Suppose also that C is described by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ on an interval $[a,b]$, and that the curve is traversed exactly once as t increases from a to b. Then,

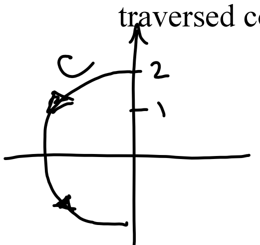
$$\int_C f(x,y,z)\,ds = \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\;dt$$
$$= \int_a^b f\big(x(t), y(t), z(t)\big)\|\mathbf{r}'(t)\|\,dt$$

Note: If $f(x,y,z) = 1$, then the line integral gives the arc length of C:

$$\text{Arc length} = \int_C 1\,ds = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\;dt = \int_a^b \|\mathbf{r}'(t)\|\,dt .$$

Note: The value of the line integral is independent of the parametrization chosen for C.

Example 3: Evaluate $\int_C (3x+4y)\,ds$, where C is the left half of the circle $x^2+y^2=4$, traversed counterclockwise.



C can be parametrized as $x = 2\cos t$, $y = 2\sin t$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

$$\int_C (3x+4y)\,ds = \int_{\pi/2}^{3\pi/2} \big[3(2\cos t) + 4(2\sin t)\big]\sqrt{(-2\sin t)^2 + (2\cos t)^2}\;dt$$

$$= \int_{\pi/2}^{3\pi/2} \big[6\cos t + 8\sin t\big]\sqrt{4(\cancel{\sin^2 t + \cos^2 t})}\;dt = \sqrt{4}\int (6\cos t + 8\sin t)\,dt$$

$$= 2\big(6\sin t - 8\cos t\big)\Big|_{\pi/2}^{3\pi/2} = 2\Big[6\sin\tfrac{3\pi}{2} - 8\cancelto{0}{\cos\tfrac{3\pi}{2}} - 6\sin\tfrac{\pi}{2} + 8\cancelto{0}{\cos\tfrac{\pi}{2}}\Big]$$

$$= 2[6(-1) - 0 - 6(1)] = 2[-12] = \boxed{-24}$$

Example 4: Evaluate $\int_C (x^2y+3xy^3)\,ds$ along the line from (0,0) to (3,9).

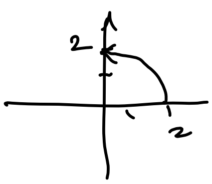
$\vec{r}(t) = \langle t, 3t\rangle$ for $0 \le t < 3$

$\|\vec{r}\,'(t)\| = \sqrt{(1)^2+(3)^2} = \sqrt{10}$

$$\int_C (x^2y+3xy^3\,ds) = \int_0^3 \left(t^2\cdot 3t + 3(t)(3t)^3\right)\|\vec{r}\,'(t)\|\,dt = \int_0^3 (3t^3+81t^4)\sqrt{10}\,dt$$

$$= \sqrt{10}\left[\frac{3t^4}{4}+\frac{81t^5}{5}\right]\Bigg|_0^3 = \sqrt{10}\left[\frac{3(3^4)}{4}+\frac{81(3)^5}{5} - 0\right] = \boxed{\frac{20643\sqrt{10}}{5}}$$

Example 5: Evaluate $\int_C (x^2+y^2)\,ds$ counterclockwise around the circle $x^2+y^2=4$ from $(2,0)$ to $(0,2)$.



$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle, \quad 0 \le t \le \frac{\pi}{2}$

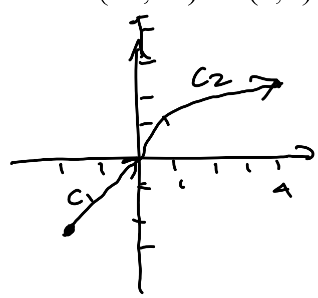
$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$

$$\int_C (x^2+y^2)\,ds = \int_0^{\pi/2} (4\cos^2 t + 4\sin^2 t)(2)\,dt$$

$$= \int_0^{\frac{\pi}{2}} 4(2)\,dt = \int_0^{\pi/2} 8\,dt = 8t\Big|_0^{\pi/2} = \frac{8\pi}{2} - 0 = \boxed{4\pi}$$

Example 6: Evaluate $\int_C 4y\,ds$, where C consists of the line segment from $(-2,-2)$ to $(0,0)$, followed by the arc of the parabola $x=y^2$ from $(0,0)$ to $(4,2)$.



Parametrize C:
$C_1: \vec{r}_1(t) = \langle t, t\rangle$ for $-2 \le t \le 0$
$C_2: \vec{r}_2(t) = \langle t^2, t\rangle$ for $0 \le t \le 2$

$$I_1 = \int_{C_1} 4y\,ds = \int_{-2}^{0} 4t\,\|\vec{r}_1{}'(t)\|\,dt$$

$$= \int_{-2}^{0} 4t\sqrt{1^2+1^2}\,dt = \int_{-2}^{0} 4\sqrt{2}\,t\,dt = 4\sqrt{2}\left(\frac{t^2}{2}\right)\Big|_{-2}^{0}$$

$$= 2\sqrt{2}\,t^2\Big|_{-2}^{0} = 2\sqrt{2}\left[0^2-(-2)^2\right] = 2\sqrt{2}(-4) = -8\sqrt{2}$$

$$I_2 = \int_{C_2} 4y\,ds = \int_0^2 4t\,\|\vec{r}_2{}'(t)\|\,dt = 4\int_0^2 t\sqrt{(2t)^2+1^2}\,dt = 4\int_0^2 t\sqrt{4t^2+1}\,dt$$

$$= 4\left(\frac{1}{8}\right)\cdot\frac{(4t^2+1)^{3/2}}{3/2}\Bigg|_0^2 = \frac{1}{2}\cdot\frac{2}{3}\left[(4(2)^2+1)^{3/2} - (4(0)^2+1)^{3/2}\right]$$

$$= \frac{1}{3}\left[17^{3/2} - 1^{3/2}\right] = \frac{17^{3/2}-1}{3}$$

$$\int_C 4y\,ds = I_1 + I_2 = \boxed{-8\sqrt{2} + \frac{17^{3/2}-1}{3}}$$

Line integrals of vector fields: Note: $ds = \|\vec r'(t)\|\,dt$

$\int_C \vec F\cdot\vec T\,ds$

$= \int_C \vec F\cdot\frac{\vec r'(t)}{\|\vec r'(t)\|}\,ds$

$= \int_C \vec F\cdot\frac{\vec r'(t)}{\cancel{\|\vec r'(t)\|}}\,\cancel{\|\vec r'(t)\|}\,dt$

$= \int_C \vec F\cdot\vec r'(t)\,dt = \int_C \vec F\cdot d\vec r$ (i.e., $\vec r'(t)\,dt = d\vec r$)

> Definition: Let **F** be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$, $a \le t \le b$. The line integral of **F** on C is
>
> $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\big(x(t), y(t), z(t)\big)\cdot\mathbf{r}'(t)\,dt.$$

An important application of this is *work* (the work done by a force in moving an object from one location to another). The work done by a force field **F** in moving an object along path C is:

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Example 7: Find the work done by $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ in moving an object along the line from $(0,0,0)$ to $(5,3,2)$.

Parametrize the path: $\vec r(t) = \langle 0,0,0\rangle + t\langle 5,3,2\rangle = \langle 5t, 3t, 2t\rangle$ for $0 \le t \le 1$

$\vec r'(t) = \langle 5,3,2\rangle$

$\|\vec r'(t)\| = \sqrt{25+9+4} = \sqrt{38}$

$\vec F(x(t), y(t), z(t)) = \langle 3t(2t), 5t(2t), 5t(3t)\rangle = \langle 6t^2, 10t, 15t^2\rangle$

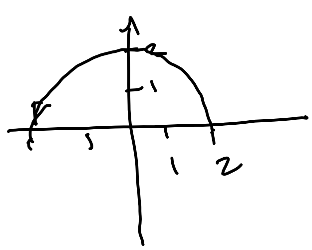
Work $= \int_C \vec F\cdot d\vec r = \int_0^1 \langle 6t^2, 10t^2, 15t^2\rangle\cdot\langle 5,3,2\rangle\,dt = \int_0^1 (30t^2 + 30t^2 + 30t^2)\,dt$

$= \int_0^1 90t^2\,dt = 90\frac{t^3}{3}\Big|_0^1 = \boxed{30}$

Example 8: Find the work done by $\mathbf{F}(x,y) = -y\mathbf{i} - x\mathbf{j}$ in moving an object counterclockwise along the semicircle $y = \sqrt{4-x^2}$ from $(2,0)$ to $(-2,0)$.

$y^2 = 4 - x^2$

Parametrize our path: $\vec r(t) = \langle t, \sqrt{4-t^2}\rangle$ for $-2 \le t \le 2$



$\vec F(x,y) = \langle -y, -x\rangle = \langle -\sqrt{4-t^2}, t\rangle$

$\vec F\cdot d\vec r = \langle -\sqrt{4-t^2}, t\rangle\cdot\langle -1, \frac12(4-t^2)^{-1/2}(-2t)\rangle$

$= \sqrt{4-t^2} - \frac{t^2}{\sqrt{4-t^2}} = \frac{4-2t^2}{\sqrt{4-t^2}}$

In trig/polar form:

$\vec r(t) = \langle 2\cos t, 2\sin t\rangle$, $0 \le t \le \pi$

$\vec F = \langle -y, -x\rangle = \langle -2\sin t, -2\cos t\rangle$, $d\vec r = \langle -2\sin t, 2\cos t\rangle dt$

$\int_0^\pi \vec F\cdot d\vec r = \int_0^\pi \langle -2\sin t, -2\cos t\rangle\cdot\langle -2\sin t, 2\cos t\rangle\,dt = \int_0^\pi (4\sin^2 t - 4\cos^2 t)\,dt$

$= \int_0^\pi -4(\cos^2 t - \sin^2 t)\,dt$

next page

$$= \int_0^{\pi} -4\cos 2t\, dt = -4\left(\frac{1}{2}\right)\sin 2t \Big|_0^{\pi}$$

$$= -2\sin\pi + 2\sin 0 = \boxed{0}$$

Line integrals in differential form:

Suppose that $\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j} + P\mathbf{k}$ and that $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. Then,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (M\,dx + N\,dy + P\,dz).$$

Example 9: Evaluate $\int_C (3y-x)\,dx + y^2\,dy$, where C is the path given by $x = 2t$, $y = 10t$, $0 \le t \le 1$.

$$\int_C \left((3y-x)\,dx + y^2\,dy\right) = \int_C (3y-x)\,dx + \int_C y^2\,dy$$

$\frac{dx}{dt} = 2$, $dx = 2\,dt$ | $\frac{dy}{dt} = 10$, $dy = 10\,dt$

$$= \int_0^1 (3(10t) - 2t)2\,dt + \int_0^1 (10t)^2\,10\,dt$$

$$= \int_0^1 (60t - 4t)\,dt + 1000\int_0^1 t^2\,dt = \int_0^1 56t\,dt + 1000\int_0^1 t^2\,dt$$

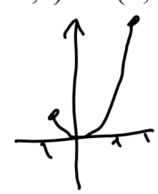
$$\frac{56}{2}t^2\Big|_0^1 + \frac{1000\,t^3}{3}\Big|_0^1 = \boxed{\frac{1084}{3}}$$

Example 10: Evaluate $\int_C xy\,dx + (x+y)\,dy$ along the curve $y = x^2$ from $(-1,1)$ to $(2,4)$.

(not done during class)

Parametrize: $\vec{r}(t) = \langle t, t^2 \rangle$

for $-1 \le t \le 2$



$x(t) = t$, $\frac{dx}{dt} = 1$, $dx = dt$ | $y(t) = t^2$, $\frac{dy}{dt} = 2t$, $dy = 2t\,dt$

$$\int_C xy\,dx + (x+y)\,dy = \int_C xy\,dx + \int_C (x+y)\,dy = \int_{-1}^2 t(t^2)\,dt + \int_{-1}^2 (t+t^2)(2t)\,dt$$

$$= \int_{-1}^2 t^3\,dt + \int_{-1}^2 (2t^2 + 2t^3)\,dt = \int_{-1}^2 (t^3 + 2t^2 + 2t^3)\,dt = \int_{-1}^2 (2t^2 + 3t^3)\,dt$$

$$= \frac{2t^3}{3} + \frac{3t^4}{4}\Big|_{-1}^2 = \frac{2(2)^3}{3} + \frac{3(2)^4}{4} - \frac{2(-1)^3}{3} - \frac{3(-1)^4}{4} = \frac{16}{3} + \frac{48}{4} + \frac{2}{3} - \frac{3}{4}$$

$$= \frac{18}{3} + 12 - \frac{3}{4} = 6 + 12 - \frac{3}{4} = 18 - \frac{3}{4} = \frac{72}{4} - \frac{3}{4} = \boxed{\frac{69}{4}}$$