15.2: Line Integrals

Recall: A curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is considered *smooth* on the interval (a,b) if $x'(t)$, $y'(t)$, and $z'(t)$ are continuous on (a,b), and if $\mathbf{r}'(t) \neq \mathbf{0}$ for every t in (a,b). (In other words, if the curve is to be smooth, then $x'(t)$, $y'(t)$, and $z'(t)$ cannot be simultaneously 0 anywhere in the interval.)

Definition: A curve C is *piecewise smooth* on $[a,b]$ if the $[a,b]$ can be partitioned into a finite number of subintervals on which C is smooth.

Example 1: Find a piecewise smooth parametrization of the path shown.

Example 2: Find a piecewise smooth parametrization of the path shown.

Line integrals:

Definition: Suppose f is defined in a region containing a smooth curve C of finite length. Suppose also that C is partitioned into n subarcs, with Δs_i representing the length of the ith subarc, and with $\|\Delta\|$ representing the length of the longest subarc. Then the line integral of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{\|\Delta\|\to 0,\, n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i\,.$$

Theorem: Evaluating a Line Integral

Suppose f is continuous in a region containing a smooth curve C. Suppose also that C is described by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ on an interval $[a,b]$, and that the curve is traversed exactly once as t increases from a to b. Then,

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\;dt \\ &= \int_a^b f\big(x(t), y(t), z(t)\big)\|\mathbf{r}'(t)\|\,dt\end{aligned}.$$

Note: If $f(x,y,z) = 1$, then the line integral gives the arc length of C:

$$\text{Arc length} = \int_C 1\,ds = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\;dt = \int_a^b \|\mathbf{r}'(t)\|\,dt\,.$$

Note: The value of the line integral is independent of the parametrization chosen for C.

Example 3: Evaluate $\int_C (3x+4y)\,ds$, where C is the left half of the circle $x^2+y^2=4$, traversed counterclockwise.

Example 4: Evaluate $\int_C (x^2 y + 3xy^3)\,ds$ along the line from (0,0) to (3,9).

Example 5: Evaluate $\int_C (x^2 + y^2)\,ds$ counterclockwise around the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$.

Example 6: Evaluate $\int_C 4y\,ds$, where C consists of the line segment from $(-2,-2)$ to $(0,0)$, followed by the arc of the parabola $x = y^2$ from $(0,0)$ to $(4,2)$.

Line integrals of vector fields:

<u>Definition</u>: Let **F** be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$, $a \le t \le b$. The line integral of **F** on C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}\big(x(t), y(t), z(t)\big) \cdot \mathbf{r}'(t)\, dt .$$

An important application of this is *work* (the work done by a force in moving an object from one location to another). The work done by a force field **F** in moving an object along path C is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} .$$

<u>Example 7:</u> Find the work done by $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ in moving an object along the line from $(0,0,0)$ to $(5,3,2)$.

<u>Example 8:</u> Find the work done by $\mathbf{F}(x, y) = -y\,\mathbf{i} - x\,\mathbf{j}$ in moving an object counterclockwise along the semicircle $y = \sqrt{4 - x^2}$ from $(2,0)$ to $(-2,0)$.

Line integrals in differential form:

Suppose that $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j} + P\mathbf{k}$ and that $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (M\,dx + N\,dy + P\,dz).$$

<u>**Example 9:**</u> Evaluate $\int_C (3y - x)\,dx + y^2\,dy$, where C is the path given by $x = 2t$, $y = 10t$, $0 \le t \le 1$.

<u>**Example 10:**</u> Evaluate $\int_C xy\,dx + (x + y)\,dy$ along the curve $y = x^2$ from $(-1,1)$ to $(2,4)$.